A Complexity Dichotomy for Permutation Pattern Matching on Grid Classes

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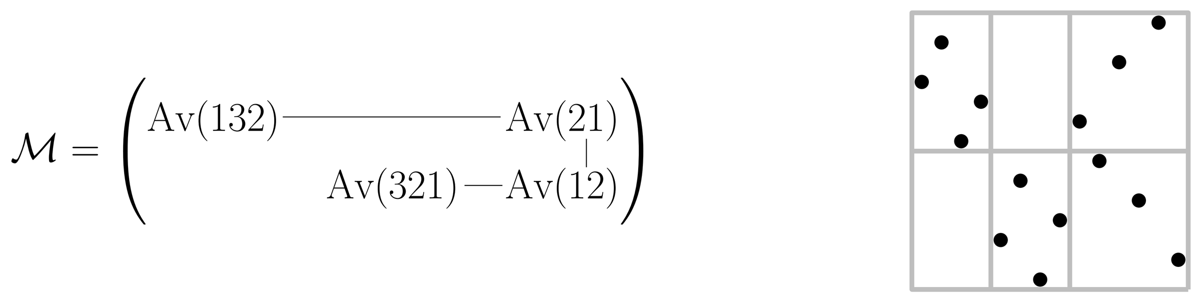
Preliminaries

The *grid class* of a matrix $\mathcal{M}$ whose entries are permutation classes, denoted by Grid($\mathcal{M}$), is a class of permutations admitting a grid-like decomposition into blocks that belong to the classes $\mathcal{M}_{i,j}$. If $\mathcal{M}$ contains only Av(21), Av(12) and $\emptyset$ we say that Grid($\mathcal{M}$) is a *monotone grid class*. To each matrix $\mathcal{M}$ we also associate a graph $G_\mathcal{M}$, called the *cell graph* of $\mathcal{M}$ whose vertices are the cells of $\mathcal{M}$ that contain an infinite class, with two vertices being adjacent if they share a row or a column of $\mathcal{M}$ and all cells between them are empty.



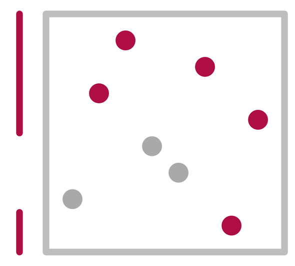
A gridding matrix $\mathcal{M}$ on the left and an $\mathcal{M}$-gridded permutation on the right. Empty entries of $\mathcal{M}$ are omitted and the edges of $G_\mathcal{M}$ are displayed inside $\mathcal{M}$.

$\mathcal{C}$-Pattern Permutation Pattern Matching ($\mathcal{C}$-Pattern PPM)
Input: A pattern $\pi \in \mathcal{C}$ of size k and a permutation τ of size n.
Question: Does τ contain π?

How hard is it to search for patterns that belong to a particular grid class, i.e. what property of $\mathcal{M}$ determines the hardness of Grid($\mathcal{M}$)-Pattern PPM ?

Grid-width

The *intervalicity* of a set $A \subseteq [n]$ is the size of the smallest interval family whose union is equal to A. For a subset S of the permutation diagram, the *grid-complexity* of S is the maximum of the intervalicities of the index set and the value set of S. For example, see a subset of grid-complexity 2 on the right.



A *grid tree* of a permutation π of length n is a rooted binary tree T with n leaves, each leaf being labeled by a distinct point of the permutation diagram. Let π_v^T denote the point set of the labels on the leaves in the subtree of T rooted in v. The *grid-with* of T is the maximum grid-complexity of the set π_v^T over all vertices v. The *grid-width* of π, denoted by gw(π), is the minimum grid-width over all possible grid trees of π.
We remark that the property of having bounded grid-width generalizes the property of having finitely many simples. The following theorem connects grid-width to PPM .

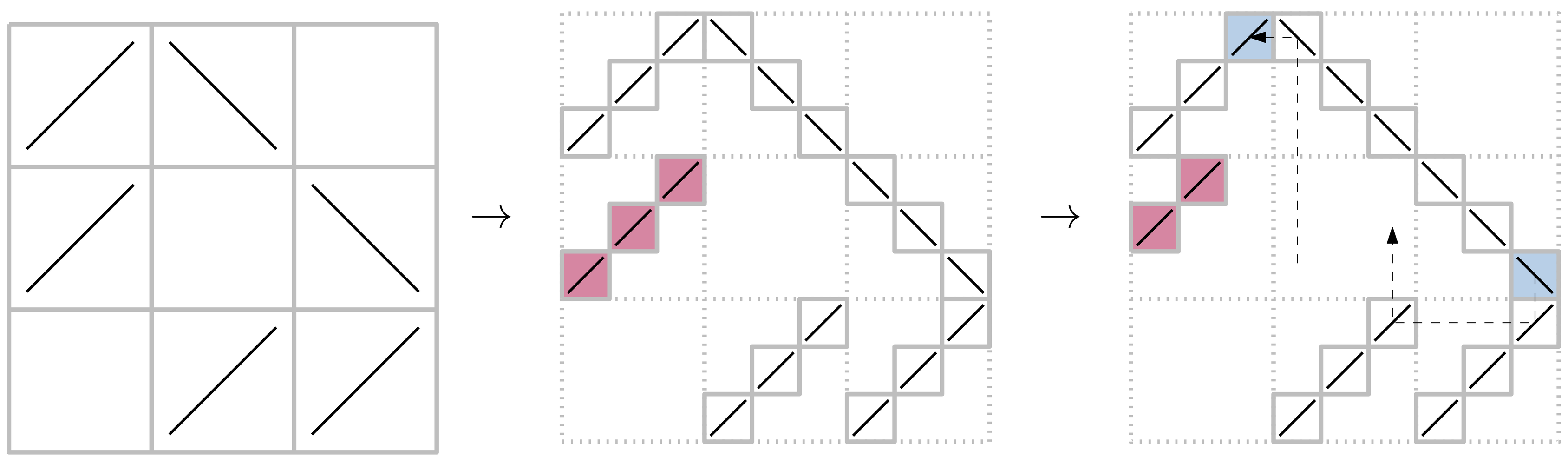
Theorem 1 **([1, 3])** *Let π be a permutation of length k and τ a permutation of length n. The problem whether τ contains π can be solved in time $n^{O(\mathrm{gw}(\pi))}$.*

The result

Let $\mathcal{M}$ be a gridding matrix whose every entry has bounded grid-width. The grid-width of Grid($\mathcal{M}$) is bounded and Grid($\mathcal{M}$)-Pattern PPM can be decided in polynomial time if $G_\mathcal{M}$ is a forest avoiding bumper-ended path. Otherwise, the grid-width of Grid($\mathcal{M}$) is unbounded and Grid($\mathcal{M}$)-Pattern PPM is NP-complete.

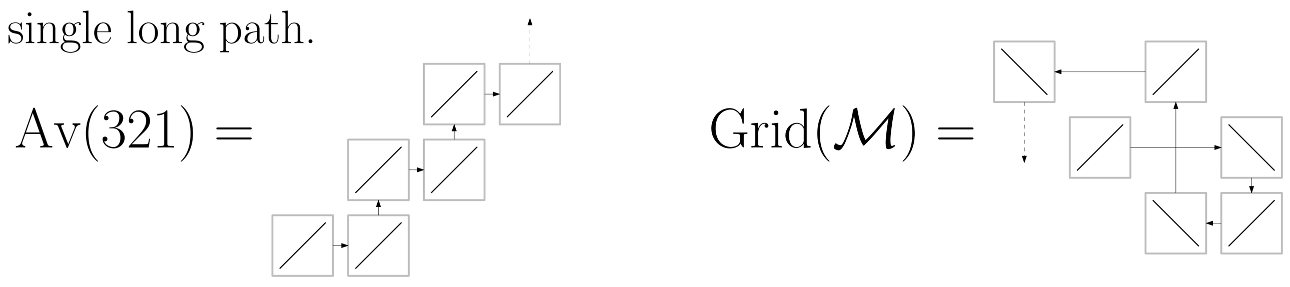
Cycle in $G_\mathcal{M}$ ⇒ arbitrarily long paths

Suppose that $G_\mathcal{M}$ contains a cycle. Then for every p, there is a gridding matrix $\mathcal{M}_p$ such that Grid($\mathcal{M}_p$) is a subclass of Grid($\mathcal{M}$) and $\mathcal{M}_p$ contains a path of length p. Idea: if we replace each cell in $\mathcal{M}$ with an appropriate $k \times k$ matrix we get $\mathcal{M}'$ such that $G_{\mathcal{M}'}$ consists of k independent copies of $G_\mathcal{M}$. Slight modification of one cell then connects the k copies of cycle into one long path. See example below.

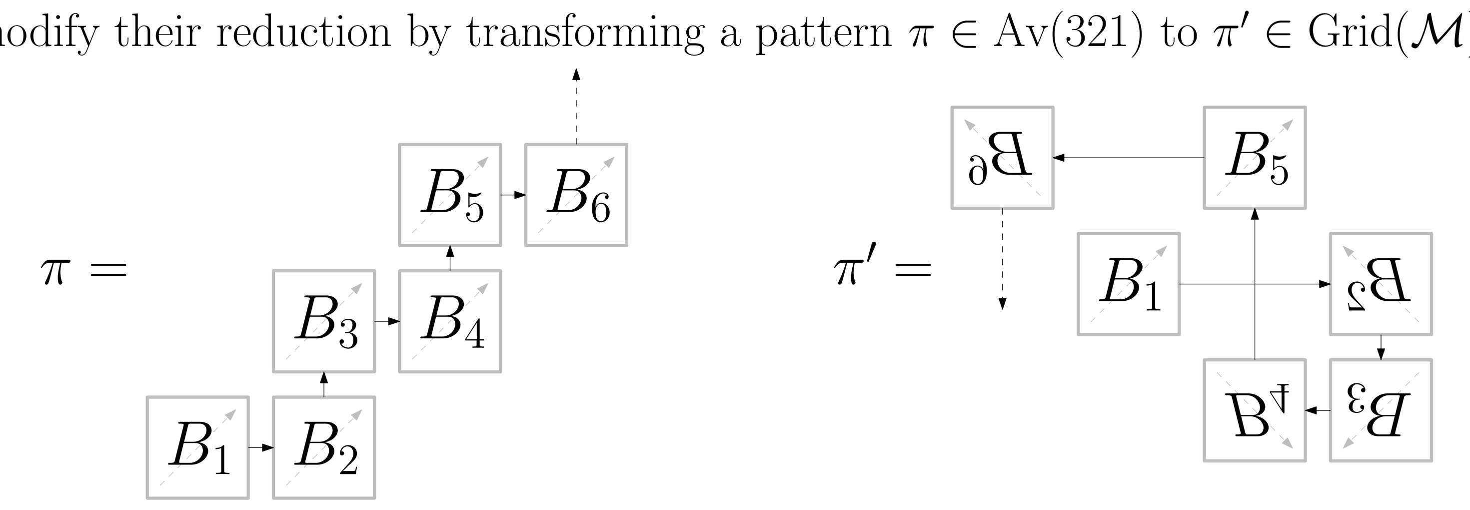


Long paths ⇒ NP-completeness

Jelínek and Kynčl [2] showed that Av(321)-Pattern PPM is NP-complete. Observe that any permutation from Av(321) in fact belongs to a monotone grid class whose cell graph is a single long path.



We modify their reduction by transforming a pattern $\pi \in$ Av(321) to $\pi' \in$ Grid($\mathcal{M}$).

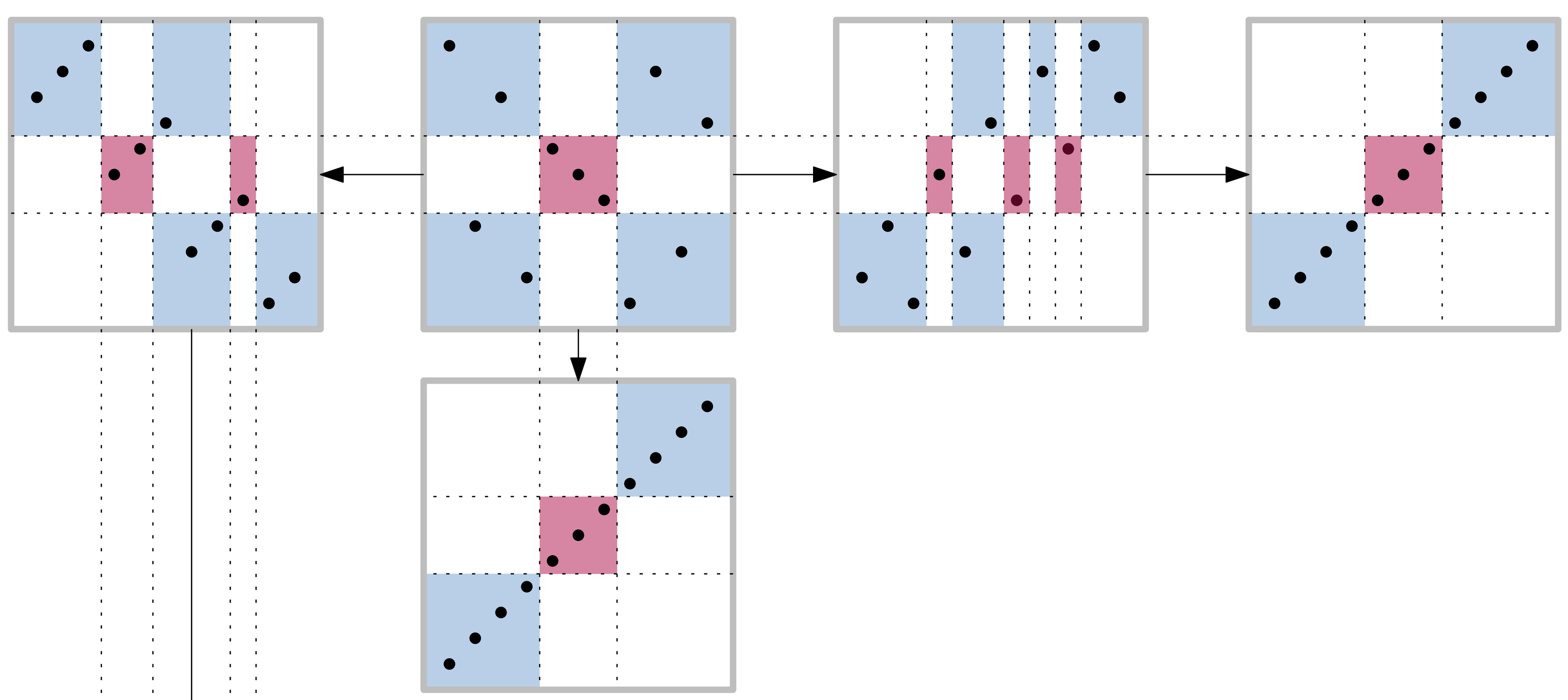


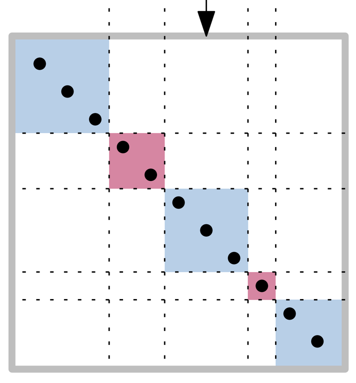
Bumper-ended paths

A *horizontal monotone juxtaposition* is a monotone grid class Grid($\mathcal{C}\ \mathcal{D}$) where both $\mathcal{C}$ and $\mathcal{D}$ are non-empty. Similarly, a *vertical monotone juxtaposition* is a monotone grid class Grid$\left(\begin{smallmatrix}\mathcal{C}\\ \mathcal{D}\end{smallmatrix}\right)$. We remark that there is a connection to insertion encodable classes – a finitely based permutation class is regular insertion encodable if and only if it does not contain any vertical monotone juxtaposition as a subclass.
An ordered pair (p, q) of vertices in $G_\mathcal{M}$ is a *bumper* if either $\mathcal{M}_q$ contains a horizontal monotone juxtaposition and shares the same column with $\mathcal{M}_p$, or if $\mathcal{M}_q$ contains a vertical monotone juxtaposition and shares the same row with $\mathcal{M}_p$. A *bumper-ended path* is a path $P = p_1, \ldots, p_k$ in $G_\mathcal{M}$ such that both (p_2, p_1) and (p_{k-1}, p_k) are bumpers.

$G_\mathcal{M}$ avoids a bumper-ended path

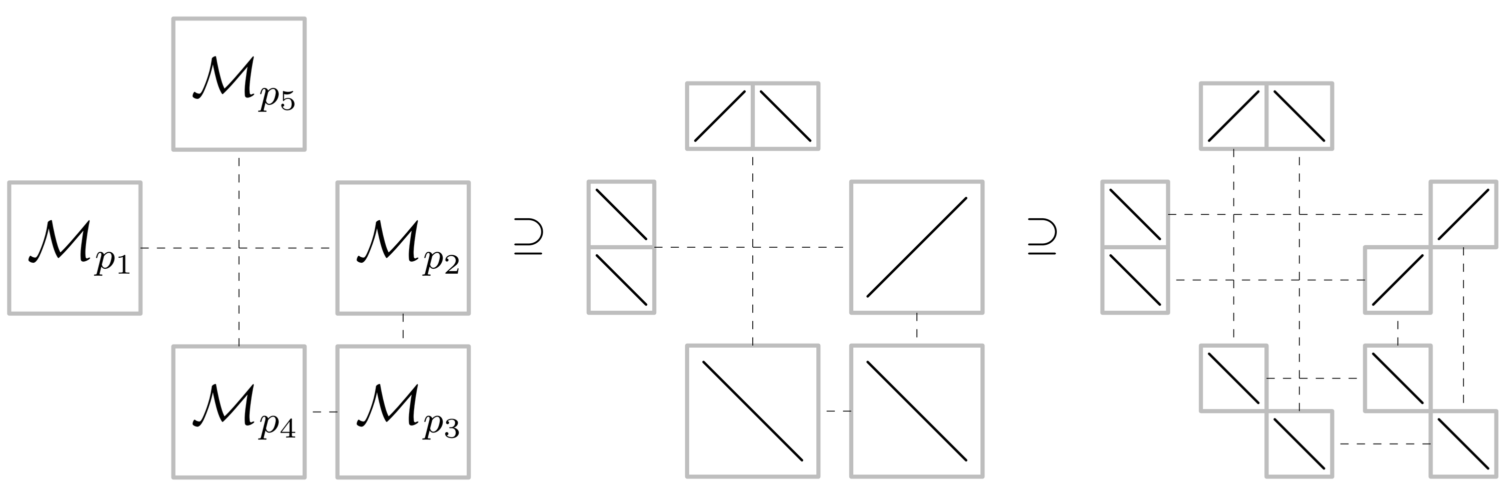
We show that Grid($\mathcal{M}$) has bounded grid-width which implies a polynomial algorithm for Grid($\mathcal{M}$)-Pattern PPM due to Theorem 1. Suppose that $G_\mathcal{M}$ is a tree. We root the tree in a way such that no path from root ends with a bumper, take an optimal grid tree of its root cell and then cut the rest of the permutation accordingly. See example below.





$G_\mathcal{M}$ contains a bumper-ended path

Let $\mathcal{M}_{p_1}, \mathcal{M}_{p_2}, \ldots \mathcal{M}_{p_k}$ be a bumper-ended path. Replace the endpoints $\mathcal{M}_{p_1}$ and $\mathcal{M}_{p_k}$ with their respective monotone juxtaposed subclasses and replace every $\mathcal{M}_{p_i}$ for i between 2 and $k-1$ with its monotone subclass. This way we obtain a monotone grid subclass that contains cycle in its cell graph.



References

[1] Ahal, S., and Rabinovich, Y. On the complexity of the sub-permutation problem. In *Algorithm theory—SWAT 2000 (Bergen)*, vol. 1851 of *Lecture Notes in Comput. Sci.* Springer, Berlin, 2000, pp. 490–503.

[2] Jelínek, V., and Kynčl, J. Hardness of permutation pattern matching. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms* (2017), SIAM, Philadelphia, PA, pp. 378–396.

[3] Jelínek, V., Opler, M., and Valtr, P. Generalized coloring of permutations. In *26th European Symposium on Algorithms*, vol. 112 of *LIPIcs. Leibniz Int. Proc. Inform.* Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018, pp. Art. No. 50, 14.