Encoding labelled p-Riordan graphs by words and pattern-avoiding permutation

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Riordan Matrices

Definition

A Riordan matrix $L = [\ell_{ij}]_{i,j\geq 0}$ generated by two formal power series

$$g = \sum_{n=0}^{\infty} g_n t^n \quad \text{and} \quad f = \sum_{n=1}^{\infty} f_n t^n \qquad \text{in } \mathbb{Z}[[t]]$$

is denoted as (g, f) and defined as an infinite lower triangular matrix whose j-th column generating function is gf^j, *i.e.* $\ell_{ij} = [t^i]gf^j$ where $[t^k]\sum_{n\geq 0} a_n t^n = a_k$.

If $g_0 \neq 0$ and $f_1 \neq 0$ then the Riordan matrix is called proper.

Riordan Graphs

Definition

A simple graph G of order n is said to be a Riordan graph if the **adjacency matrix** $A(G)$ can be expressed as

$$A(G) \equiv (tg, f)_n + (tg, f)_n^T \pmod{2}$$

for some generating functions g and f over $\mathbb{Z}$ where $(tg, f)_n$ is the $n \times n$ leading principle matrix of the Riordan matrix (tg, f).

A Riordan graph G on n vertices with the adjacency matrix $A(G)$ given by above is denoted as $G = G_n(g, f)$. Moreover, we denote the set of Riordan graphs with n vertices by $\mathcal{RG}_n$.

- The number r_n of Riordan graphs of order $n \geq 1$ is $r_n = \frac{4^{n-1}+2}{3}$.

Riordan Graphs

Example (Pascal matrix)

Let $g = \frac{1}{1-t}$ and $f = \frac{t}{1-t}$, then

$$gf^j = \frac{1}{1-t}\left(\frac{t}{1-t}\right)^j = \frac{t^j}{(1-t)^{j+1}} = \sum_{i=0}^{\infty}\binom{i}{j}t^i \qquad , j \geq 0.$$

So, the Riordan matrix (g, f) is $\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & O & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

Riordan Graphs

Example (Pascal matrix continued)

The adjacency matrix of $G_6(g, f)$ is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

p-Riordan Graphs

Definition

The definition of a p-Riordan graph is obtained by replacing "mod 2" by "mod p" in the definition of a Riordan graph where p is a **prime** number.
We denote the set of p-Riordan graphs with n vertices by $\mathcal{RG}_n^{(p)}$.
In particular, the case of $p = 3$ corresponds to oriented Riordan graphs.

- the number $r_n^{(p)}$ of p-Riordan graphs for a prime p is $r_n^{(p)} = \frac{p^{2(n-1)}+p}{p+1}$.

Encoding p-Riordan Graphs by Words

Definition

Let $\mathcal{W}_n^{(p)}$ be the set of words $w = w_1 w_2 \cdots w_n$, $n \geq 1$, over the alphabet $A^{(p)} = \{a_{i,j} : 0 \leq i, j \leq p-1\}$ such that either

- $w = a_{0,0} a_{0,0} \cdots a_{0,0}$, or
- there exist i and b, $1 \leq i \leq n$, $1 \leq b \leq p-1$, such that $w_1 \cdots w_i = a_{0,0} a_{0,0} \cdots a_{0,0} a_{b,0}$.

Theorem 1

There is a bijection between $\mathcal{RG}_{n+1}^{(p)}$ and $\mathcal{W}_n^{(p)}$.

- Theorem 1 is the reason we call the words in $\mathcal{W}_n^{(p)}$ p-Riordan words.

Pattern-Avoiding Permutations

Definition

The reduced form of a permutation π, denoted by red(π), is the permutation obtained from π by substituting the i-th smallest element by i. For example, red(4287)=2143.

We say that a permutation $\pi = \pi_1 \cdots \pi_n$ avoids a pattern $\tau = \tau_1 \cdots \tau_k$ if there are indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\text{red}(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}) = \tau$.

Notation

The set of n-permutations that avoid simultaneously the patterns 123 and 132 is denoted by $S_n(123, 132)$.

- $|S_n(123, 132)| = 2^{n-1}$

Encoding Riordan Graphs by Pattern-Avoiding Permutations

- Let P_{2n} be the set of permutations in $S_{2n}(123, 132)$ with **two fixed points**.

Theorem 2

There is a bijection between $\mathcal{RG}_n$ and P_{2n}.

- Let $\mathcal{B}_n$ be the set of words of length $2n$ on alphabet $\{0, 1, 2\}$ with an even number (possibly zero) of each letter. We call these words balanced words. **Balanced words** are connected to **oriented Riordan graphs** via the following theorem.

Theorem 3

There is a bijection between $\mathcal{RG}^{(3)}_{n+1}$ and $\mathcal{B}_n$.

Open problems

- Explain combinatorially that Riordan graphs in $\mathcal{RG}_n$ are in one-to-one correspondence with permutations in $S_n(4321, 4123)$, or in $S_n(4321, 3412)$, or in $S_n(4123, 3214)$, or in $S_n(4123, 2143)$ that were enumerated in [D. Kremer, W. C. Shiu. Finite transition matrices for permutations avoiding pairs of length four patterns. *Discrete Appl. Math.* **268** (2003) 171–183.]; also see A047849 in the On-Line Encyclopedia of Integer Sequences.