k-Pop Stack Sortable Permutations

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Permutation Patterns 2020

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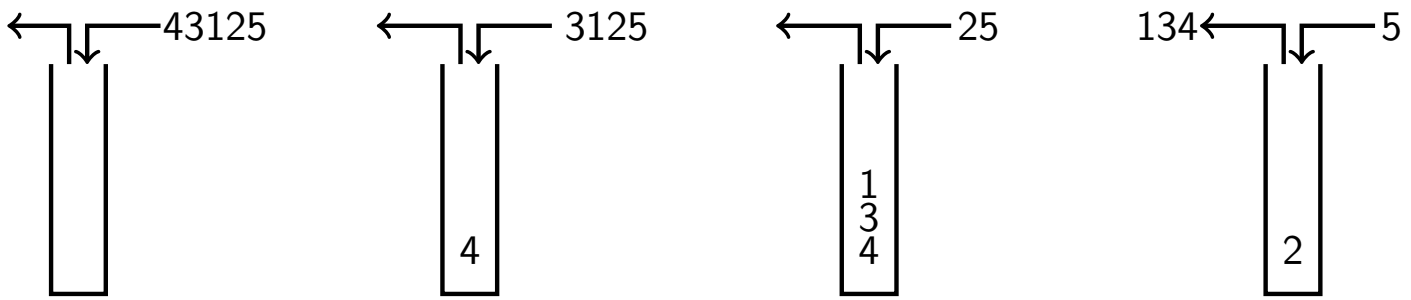
Eg: input 43125



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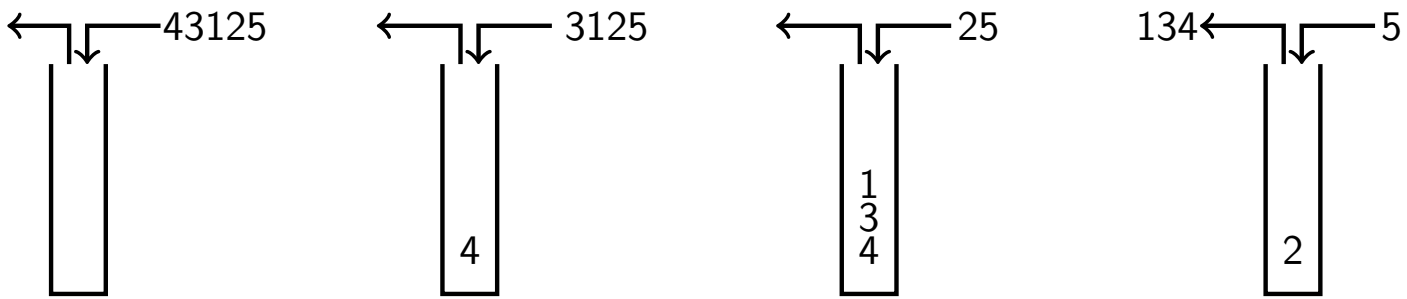

output: 13425

k-pop stack sortable permutation

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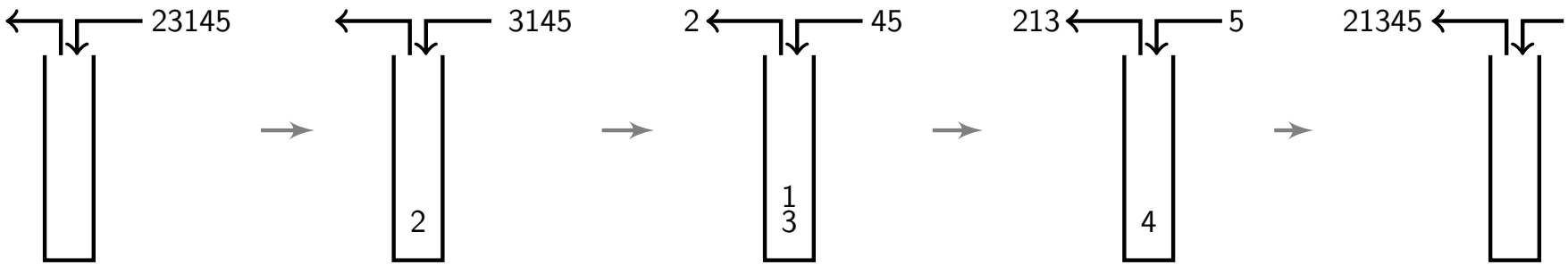

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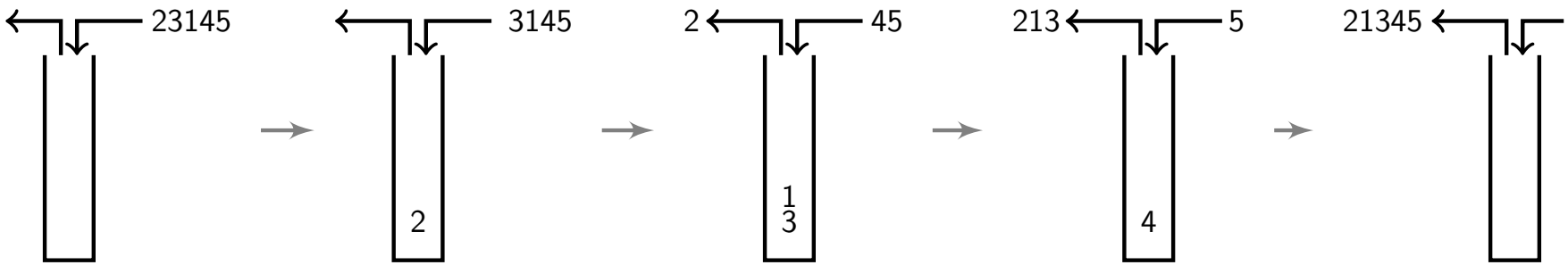

2-pass:

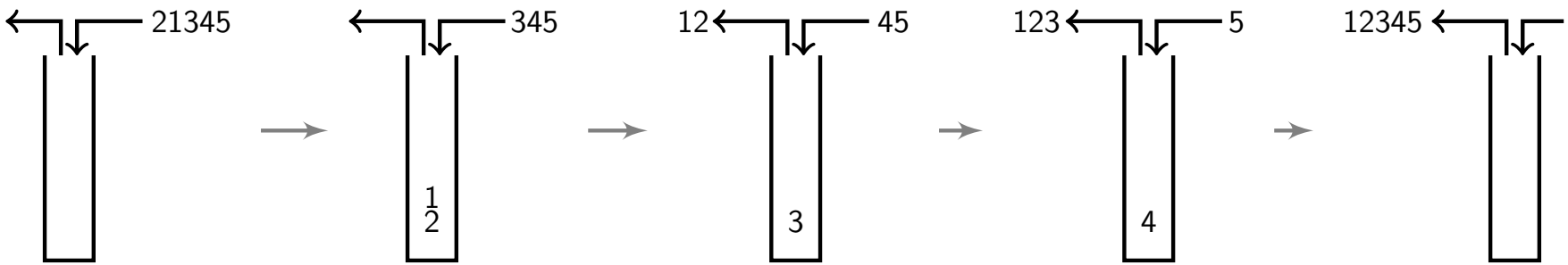

Figure: Sorting 23145 with a 2-pass pop stack

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Question (Claesson and Guðmundsson [2])

Is there a "useful permutation pattern characterization of the k-pop stack-sortable permutations?"

Some notation

1. $\gamma <_{\text{subperm}} \alpha$: γ is a subpermutation of α.
2. red(γ) : permutation obtained by replacing the tth smallest entry of γ by the integer i.
3. Let σ be a permutation. Call a factor $B_i = a_{i,1}a_{i,2}\ldots a_{i,n_i}$ of σ a **block** if $n_i > 0$ and $a_{i,j} > a_{i,j+1}$ for all $1 \leqslant j < n_i$. (a factor means the entries are contiguous in σ.)

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2-avoidance

Let σ be a permutation and $F, G \subseteq S^\infty$ be two sets of reduced permutations. We say that σ 2-*avoids* (F, G) if for all $\gamma <_{\text{subperm}} \sigma$, if $\text{red}(\gamma) \in F$ then there exists $\delta <_{\text{subperm}} \sigma$ such that $\gamma <_{\text{subperm}} \delta$ and $\text{red}(\delta) \in G$.

We denote the set of all permutations in S^∞ which 2-avoid (F, G) by $\text{Av}_2(F, G)$.

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Theorem (E, Goh)

Let $k \in \mathbb{N}_+$. There exists a pair of finite sets (F_k, G_k) such that the set of all k-pass pop stack sortable permutations is equal to $\mathrm{Av}_2(F_k, G_k)$. Moreover, the sets F_k, G_k can be algorithmically constructed.

Let S_k denote the set of all k-pass pop stack sortable permutations. We proceed by induction, with the base case $k = 1$ established by Avis and Newborn [1] (specifically, $F_1 = \{231, 312\}, G_1 = \emptyset$).

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Let $f_{\max} = \max\{|\beta| \mid \beta \in F_{k-1}\}$ and $C = 3^{k+2} f_{\max}$. Then define

$$\begin{aligned} \Omega_1 &= \{\tau \in S^\infty \mid |\tau| \leqslant 3f_{\max}, \tau \notin S_k\}, \\ \Omega_2 &= \{\kappa \in S^\infty \mid |\kappa| \leqslant C, \kappa \in S_k, \exists \tau \in \Omega_1[\tau \leqslant \kappa]\}. \end{aligned}$$

1. Claim 1: Ω_i are both finite: both are subsets of the set of all permutations of length at most C.
2. Claim 2: Ω_i are algorithmically constructible: There is only finitely many τ, κ of length at most $3f_{\max}, C$ respectively.
3. Claim 3: $\sigma \notin S_k$ if and only if σ 2-contains (Ω_1, Ω_2): this is to check the closure properties such that everything not sortable by k-pop stack must 2-contains (Ω_1, Ω_2).
    1. We proved that $p_1(\sigma)$ 2-contains (F_{k-1}, G_{k-1}) which implies $p_1(\sigma) \notin S_{k-1}$ which implies $\sigma \notin S_k$.
    2. For contradiction, we assume $\sigma \in S_k$. Then, we proved that $\kappa <_{\text{subperm}} \sigma$ such that $\kappa \in S_k, \gamma <_{\text{subperm}} \kappa$ and $|\kappa| \leqslant C$ can be constructed. After that, can construct $\alpha \in \Omega_2$ with $\alpha \sim \kappa$ and $\text{red}(\gamma) \leqslant \alpha$, which means α saves[1] $\gamma <_{\text{subperm}} \sigma$, and this gives a contradiction that σ 2-contains (Ω_1, Ω_2) because of γ.

[1] refer to Murray' talk

Construction of κ

1. Starting with $\kappa{=}\sigma$, mark the tokens corresponding to γ **bold**.
2. Delete non-bold tokens given that no blocks in k are merged.
3. At the end of this process, each block in k contains at most two non-bold entries. See Figure 2.

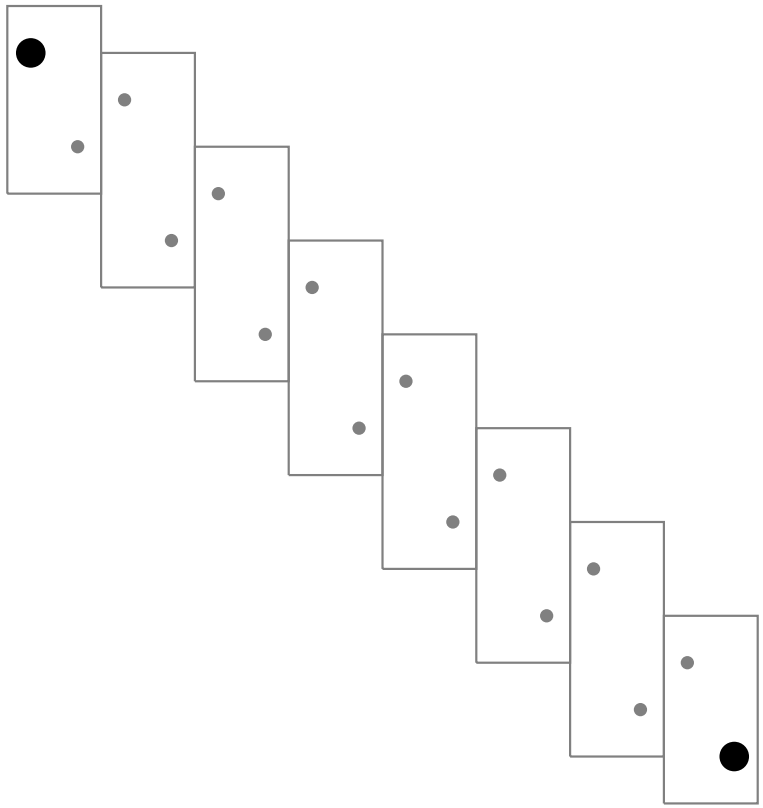

Figure: $\kappa = \mathbf{16}, 14, 15, 12, 13, 10, 11, 896745231$

Lemma

Let σ be a permutation with block decomposition $B_1 B_2 B_3 \dots B_m$, $a \in B_{i+1}, b \in B_{i+n}$ two entries of σ with $a > b$, $n \geqslant 1$. (See for example Figure 2). If $n \geqslant 3^k$ then σ is not k-pass pop stack sortable.

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If κ cannot be sorted, then $p_1(\kappa) \notin S_{k-1}$ because of some subpermutation τ with $\mathrm{red}(\tau) \in F_{k-1}$, but since no block has merged in obtaining $p_1(\kappa)$, $p_1(\sigma)$ also contains τ which cannot be saved since blocks containing the tokens forming τ are fixed. Thus $p_1(\sigma) \notin S_{k-1}$, so $\sigma \notin S_k$, contradiction.

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Worst case:

$$|\kappa| \leqslant (3 + 2.3^k)|\gamma| \leqslant (3 + 2.3^k)3f_{\max} \leqslant 3^{k+2} f_{\max} = C$$

Conclusion

Our proof shows that pattern characterization of k-pop stack sortable permutation is finite, but does not give the smallest sets of patterns. This is because the number of elements to be checked is $C!$ (upper bound). Recall that $C = 3^{k+2} f_{\max}$.

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However, this upper bound can certainly be lowered with some lemmas. The detail of the lemmas can be seen in this paper [3].

Thank you!

References

D. Avis and M. Newborn.

On pop-stacks in series.

Utilitas Math., 19:129–140, 1981.

A. Claesson and B. Ágúst Guðmundsson.

Enumerating permutations sortable by k passes through a pop-stack.

Sém. Lothar. Combin., 80B:Art. 43, 12, 2018.

M. Elder and Y. K. Goh.

k-pop stack sortable permutations and 2-avoidance.

arXiv e-prints, Nov. 2019.

L. Pudwell and R. Smith.

Two-stack-sorting with pop stacks.

Australasian Journal of Combinatorics, 74(1):179 – 195, 2019.