Preimages under Queuesort

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Synopsis

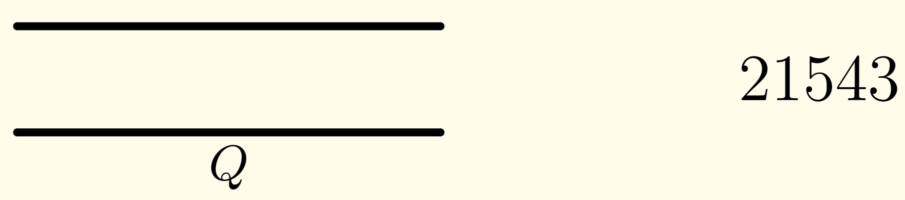
Queuesort is an algorithm which tries to sort an input permutation π by using a queue. Similarly to what has been done for Stacksort (more recently by Defant and less recently by Bousquet-Mélou), we study preimages of permutations under Queuesort.

Characterization of preimages. Alternative way of describing the output $q(\pi)$ of Queuesort on input π. Recursive characterization of preimages of a given permutation, from which we deduce a recursive procedure to generate them.

Enumeration of preimages. Enumeration of $q^{-1}(M_1P_1M_2)$, where M_1, M_2 are all the maximal sequences of LTR maxima of π (with M_2 possibly empty): combinations of Catalan numbers. Catalan numbers when $|M_2| = 1$. Enumeration of permutations with one preimage: derangement numbers.

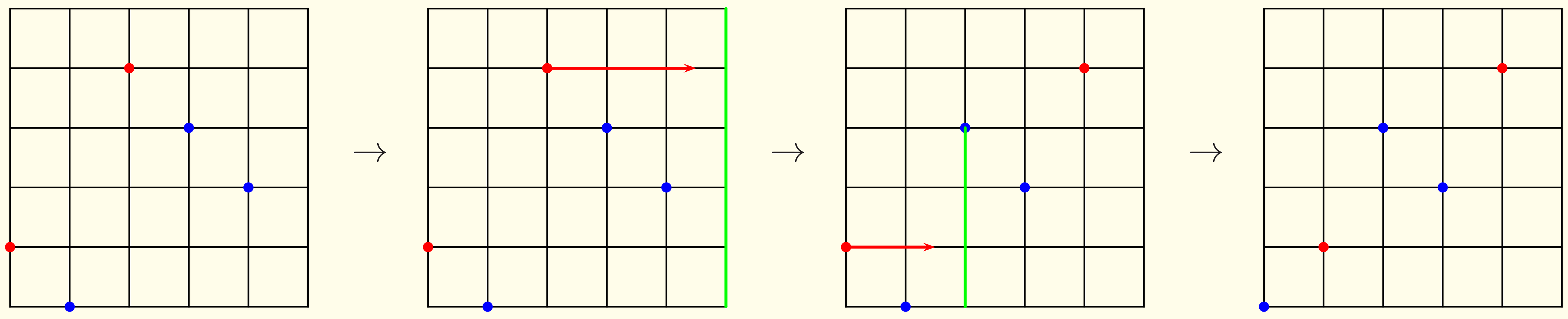
The algorithm Queuesort

```
for i=1 to n do
begin
if (Q=∅ or BACK(Q)<π_i) then ENQUEUE
else
begin
while FRONT(Q)<π_i do DEQUEUE
OUT
end
end
while Q≠∅ do DEQUEUE
```



Description of Queuesort on permutations

Read the permutation from right to left, and move each $LTR-maxima$ to the right till we find a number bigger than it.



Preimage not empty

Let $\pi = \pi_1 \cdots \pi_n$. Then $q^{-1}(\pi) \neq \emptyset$ if and only if $\pi_n = n$.

Notation

$\pi = M_1P_1 \cdots M_{k-1}P_{k-1}M_k$, in which the M_i's are all the maximal, nonempty sequences of consecutive $LTR-maxima$. Sometimes we will use N_i and R_j instead of M_i and P_j, respectively.

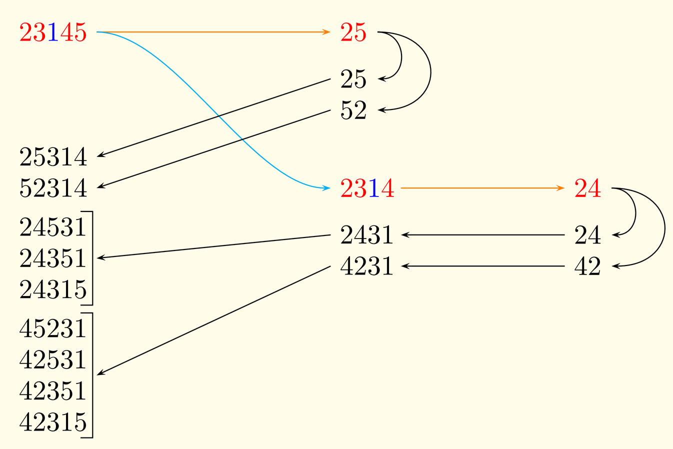
Characterization of preimages

If π is not the identity permutation, then every preimage σ of π is one and only one of the following:

- $\sigma = q^{-1}(M_1P_1 \cdots M_{k-2}P_{k-2}M'_{k-1}n)m_{k-1,m_{k-1}}P_{k-1}M'_k$, where M'_{k-1} is obtained by removing the last element $m_{k-1,m_{k-1}}$ from M_{k-1}, while M'_k is obtained by removing n from M_k;
- let π' be the permutation obtained by removing n from π; for every preimage $\sigma' = N_1R_1 \cdots N_{s-1}R_{s-1}N_s$ of π', σ is obtained by inserting n in one of the positions to the right of N_{s-1}.

Associated algorithm

- compute all the preimages of $M_1P_1 \cdots M_{k-2}P_{k-2}M'_{k-1}n$, and concatenate them with $m_{k-1,m_{k-1}}P_{k-1}M'_k$;
- if $|M_k| \geq 2$, then compute a preimage $\sigma' = N_1R_1 \cdots N_{s-1}R_{s-1}N_s$ of $M_1P_1 \cdots M_{k-1}P_{k-1}M'_k$, and insert n in each of the positions to the right of N_{s-1}.



Remark

The number of preimages is uniquely determined by the positions of its $LTR-maxima$.

Permutations with one preimage

A permutation $\pi \in S_n$ has exactly one preimage if and only if it ends with n and has no adjacent $LTR-maxima$. As a consequence, using Foata's fundamental bijection, $|\{\pi \in S_n \mid |q^{-1}(\pi)| = 1\}|$ is the number of derangements of length $n-1$.

Preimages of M_1P_1n

Let $\pi = M_1P_1M_2$, with $|M_2| = 1$. Then $q^{-1}(\pi) = |C_{m_1}|$, where C_k is the k-th Catalan number.

Preimages of $\pi = M_1P_1M_2$

Let $\pi = M_1P_1M_2 \in S_n$. Then

$$|q^{-1}(\pi)| = \sum_{i=1}^{m_2}\sum_{j=0}^{i-1}\binom{i-1}{j}\left(\sum_{l=0}^{m_1-2}\left(\!\binom{m_1-l}{j+1}\!\right)b_{m_1-1,l+1}\right)\left(\sum_{k=2}^{m_2-i+1}g_{m_2-i,k}\left(\!\binom{k}{p_1+i-j-1}\!\right)\right)$$

where $b_{r,s} = A009766(r,s)$, and $g_{r,s} = A033184(r,s-1)$ are two different incarnations of the ballot numbers.
In the above formula, the sum involving the $b_{r,s}$'s can be rewritten as

$$\sum_{l=0}^{m_1-2}\left(\!\binom{m_1-l}{j+1}\!\right)b_{m_1-1,l+1} = \sum_{i=1}^{\lfloor\frac{j+1}{2}\rfloor+1}(-1)^{i-1}\binom{j+2-i}{i-1}C_{m_1+j+1-i}.$$

This shows that $|q^{-1}(\pi)|$ can be expressed as a combination of Catalan numbers.