Permutations sortable by two restricted stacks: Catalan and Schröder cases

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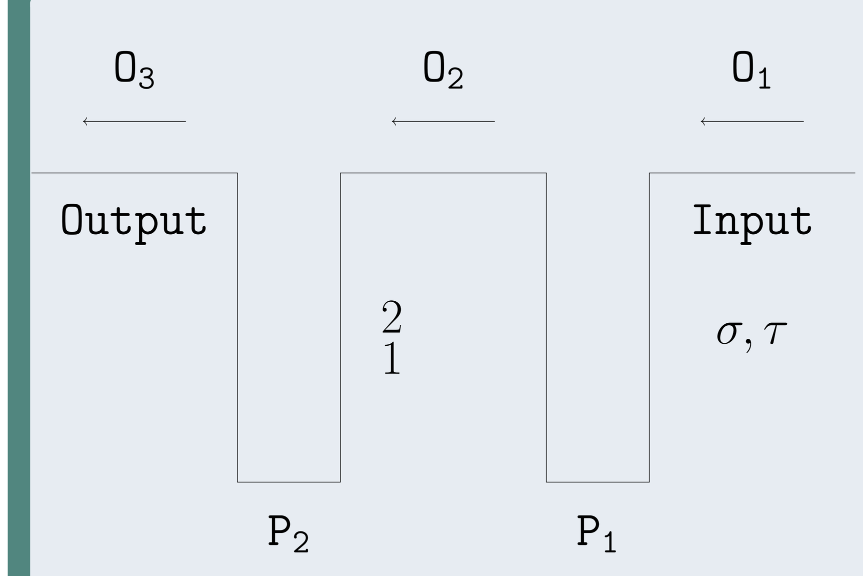
The (σ,τ)-machine



Restrictions:

- Right-greedy algorithm;
- P_1 avoids σ and τ (from top to bottom);
- P_2 avoids 21.

Operations Priority:

(i) O_1 pushes the next element of the input into P_1;

(ii) O_2 pops the top of P_1 and pushes it into P_2;

(iii) O_3 pops the top of P_2 and pushes it into the output.

Example: If $\sigma = 123$, $\tau = 132$ then 35124 is sortable by the sequence

$$O_1, O_1, O_2, O_1, O_1, O_1, O_2, O_2, O_2, O_3, O_3, O_2, O_3, O_3, O_3.$$

Enumerative results for $|\mathrm{Sort}(\sigma,\tau)|$

σ,τ	Sequence $\lvert Sort_n(\sigma,\tau)\rvert$	Avoided patterns	OEIS
123,132	Catalan $1, 2, 5, 14, 42, 132, 429, \ldots$	3214, 2314, 4213, $[24\bar{1}3$	A000108
123,213		Conjecture: $[24\bar{1}3$, $[4231$, $[31425$, $[421\bar{3}5$, mesh 2413	
132,312	Catalan $1, 2, 5, 14, 42, 132, 429, \ldots$	?	A000108
231,321		?	
132,231	Large Schröder $1, 2, 6, 22, 90, 394, 1806, \ldots$	1324, 2314	A006318

Definitions

- π is (σ,τ)-sortable if the output of P_2 is the identity;
- $\mathrm{Sort}(\sigma,\tau)$ set of (σ,τ)-sortable permutations;
- $\mathcal{S}^{\sigma,\tau}(\pi)$ output of P_1 on input π.

Previous Works

- *Stack sorting with restricted stacks*, G. Cerbai, A. Claesson, L. Ferrari, JCTA, 2020.
- *Sorting Cayley permutations with pattern-avoiding machines*, G. Cerbai, arXiv:2003.02536, 2020.
- *Catalan and Schröder permutations sortable by two restricted stacks*, J.-L. Baril, C. Khalil, V. Vajnovszki, arXiv:2004.01812, 2020
- *The 132-avoiding machine*, G. Cerbai, A. Claesson, L. Ferrari, E. Steingrimsson, arXiv:2006.05692, 2020.

Future work

- Characterize and enumerate $\mathrm{Sort}(123,321)$ and $\mathrm{Sort}(132,321)$.
- Prove that $\mathrm{Sort}(123,312)$ is enumerated by the binomial transform of the Catalan numbers A007317.
- Sorted permutations and fertilites for the (σ,τ)-machine.

The $(123,132)$-machine

Theorem 1

$$\mathrm{Sort}(123,132) = \mathrm{Av}\left(3214, 2314, 4213, [24\bar{1}3\right).$$

Theorem 2

Let $\mathrm{Sort}_n^k = \{\pi \in \mathrm{Sort}_n(123,132) : \pi_1 = k\}$. Then:

$$|\mathrm{Sort}_n^k| = T_{n,k},$$

where $T_{n,k}$ is the (n,k)-coefficient of the Catalan triangle A009766.

Corollary

$|\mathrm{Sort}_n(123,132)| = \mathfrak{c}_n$ the n-th Catalan number.

Catalan's triangle $T_{n,k}$

$k\backslash n$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2		1	2	3	4	5	6	7
3			2	5	9	14	20	27
4				5	14	28	48	75
5					14	42	90	165
6						42	132	297
$\ldots$							$\ldots$	$\ldots$
$\sum$	1	2	5	14	42	132	429	1430

The Catalan's triangle is defined recursively by

$$T_{n,k} = T_{n,k-1} + T_{n-1,k}.$$

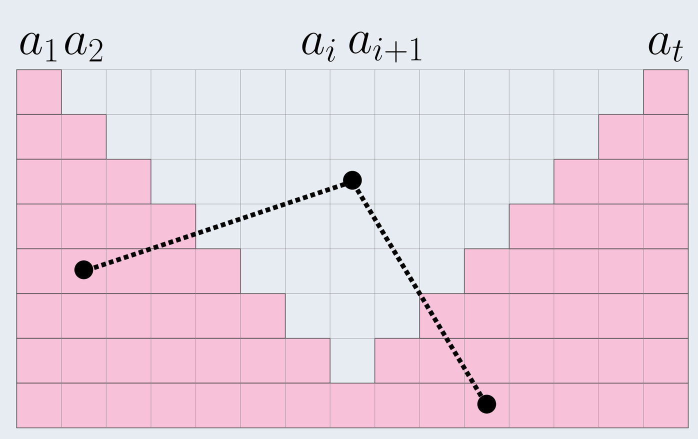
The $(132,231)$-machine

Theorem 3

$$\mathrm{Sort}(132,231) = \mathrm{Av}(1324, 2314)$$

Corollary

$|\mathrm{Sort}_n(132,231)| = \mathfrak{s}_n$ the n-th large Schröder number.



P_1 contains an anti-unimodal sequence:

$$a_1 > a_2 > \cdots > a_i < a_{i+1} < \cdots < a_t$$

The $(\sigma,\hat{\sigma})$-machine

- Let $\hat{\sigma} = \sigma_2\sigma_1\sigma_3\cdots\sigma_k$.
- Let $\mathcal{R} : S_n \to S_n$ the reverse operator.
- Let $\mathcal{S}^{\sigma,\hat{\sigma}} : S_n \to S_n$ the $(\sigma,\hat{\sigma})$-stack operator.

Theorem 4

$\mathcal{S}^{(\sigma,\hat{\sigma})}$ is bijective on S_n

$$\Longrightarrow \begin{cases} \mathrm{Sort}(\sigma,\hat{\sigma}) = \mathcal{R} \circ \mathcal{S}^{(\sigma,\hat{\sigma})} \circ \mathcal{R}(\mathrm{Av}(231)). \\ |\mathrm{Sort}_n(\sigma,\hat{\sigma})| = |\mathrm{Av}_n(231)|. \end{cases}$$