Longest increasing subsequences in pattern-avoiding permutations

Gökhan Yıldırım

Bilkent University

joint work with Toufik Mansour (University of Haifa)

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Longest increasing subsequences in permutations

The longest increasing subsequence problem for uniformly random permutations has a long and interesting history. The research around this problem, also called **Ulam's problem**, has made surprising connections among different fields such as combinatorics, probability, statistical physics and computer science.

We use one-line notation for permutations $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$.

S_n denotes the set of all permutations of length n.

For $\sigma \in S_n$, we say that $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_l}$ is an **increasing subsequence of length** l in σ if $i_1 < i_2 < \cdots < i_l$ and $\sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_l}$.

Let

$\mathrm{LIS}_n(\sigma) =$ the length of the **longest increasing** subsequence in σ.

$\mathrm{LDS}_n(\sigma) =$ the length of the **longest decreasing** subsequence in σ.

A brief history of Ulam's problem

Erdös-Szekeres lemma (1935):

$$\text{For every } \sigma \in S_n, \quad \mathsf{LIS}_n(\sigma)\,\mathsf{LDS}_n(\sigma) \geq n$$

In 1961, on the basis of some simulations, **Ulam** conjectured that

$$c_o := \lim_{n\to\infty} \frac{\mathbb{E}(\mathsf{LIS}_n)}{\sqrt{n}}$$

exists.

Hammersley (1972) proved the **existence of the limit**.

Logan and Shepp (1977), Vershik and Kerov (1977) **determined the constant:** $c_o = 2$.

Tracy-Widom limit

A real breakthrough was achieved by Baik, Deift and Johansson in 1999 by completely determining the **asymptotic distribution** of LIS_n .

Theorem (Baik-Deift-Johanson (1999))

Consider S_n with the uniform probability distribution. Then

$$\lim_{n\to\infty} Pr\left(\frac{\mathrm{LIS}_n - 2\sqrt{n}}{n^{1/6}} \leq t\right) = F_{GUE}(t) \quad \text{for all} \quad t \in \mathbb{R},$$

where F_{GUE} is the Tracy-Widom GUE distribution function.

The Tracy-Widom GUE is the largest eigenvalue distribution for the Gaussian Unitary Ensemble from random matrix theory.

An interacting particle process on $[0, 1]$

- Initially there are zero particles in the system.

- At each step, a particle appears at a uniform random point u in the interval $[0, 1]$; simultaneously the nearest particle (if any) to the right of u disappears.

Let $\mathcal{P}_n =$ the number of particles in the system after n steps.

Random variables LIS_n and $\mathcal{P}_n$ have the same probability distribution.

This model also gives a very efficient algorithm to simulate LIS_n on S_n under the uniform probability distribution.

LIS_n on pattern-avoiding permutations

Let $S_n(\tau)=$ the set of τ-avoiding permutations of length n.

some bijections on S_n: complement, reverse, inverse

Let $\sigma=\sigma_1\sigma_2\cdots\sigma_n\in S_n$.

- the **complement** of σ is $\sigma^c=\sigma_1^c\sigma_2^c\cdots\sigma_n^c$ where $\sigma_i^c=n+1-\sigma_i$ for $i\in[n]$.
- the **reverse** of σ is defined to be $\sigma^r=\sigma_n\sigma_{n-1}\cdots\sigma_1$.
- τ^{-1} is the **inverse** permutation of τ.

We have

$$|S_n(\tau)|=|S_n(\tau^c)|=|S_n(\tau^r)|=|S_n(\tau^{-1})|.$$

Note that

$$\mathrm{LIS}_n(\sigma)=\mathrm{LIS}_n(\sigma^{rc})=\mathrm{LIS}_n(\sigma^{-1})=\mathrm{LIS}_n((\sigma^{rc})^{-1})$$

and

$$\mathrm{LIS}_n(\sigma)=\mathrm{LDS}_n(\sigma^r)$$

First case: LIS_n on $S_n(\tau)$ where $\tau \in S_3$

Theorem (Deutsch-Hildebrand-Wilf (2003))

a- *In the case* $\tau = 231$, *for all* $t \in \mathbb{R}$

$$\lim_{n\to\infty} P_n^{231}\left(\frac{\mathrm{LIS}_n - (n+1)/2}{\sqrt{n}/2} \le t\right) = \Phi(t), \ \textit{standard normal}$$

b- *In the case* $\tau = 132$,

$$\lim_{n\to\infty} P_n^{132}\left(\frac{\mathrm{LIS}_n - \sqrt{\pi n}}{\sqrt{n}} \le t\right) = \sum_{s=-\infty}^{\infty} (1 - 2s^2(t + \sqrt{\pi})^2) e^{-(t+\sqrt{\pi})^2 s^2}$$

for $t > -\sqrt{\pi}$.

c- *In the case* $\tau = 321$, *for all* $t \in \mathbb{R}$

$$\lim_{n\to\infty} P_n^{321}\left(\frac{\mathrm{LIS}_n - n/2}{\sqrt{n}} \le t\right) = \frac{2}{\sqrt{\pi}} \int_0^{4t^2} u^{1/2} e^{-u} du$$

Second case: LIS_n on $S_n(\tau^1, \tau^2)$ where $\tau^1, \tau^2 \in S_3$

Theorem (Madras - Y. (2017))

Consider $S_n(\tau^1, \tau^2)$ with uniform probability distribution where $\tau^1, \tau^2 \in S_3$. Then we have:

	LIS_n			LDS_n		
$\{\tau_1, \tau_2\}$	*mean*	*SD*	*asymptotically normal?*	*mean*	*SD*	*asymptotically normal?*
$\{132, 321\}$	$\sim \frac{5n}{6}$	$\sim \frac{5n}{6\sqrt{2}}$	*No*	$\to 2$	$\to 0$	*No*
$\{132, 231\}$	$\frac{n+1}{2}$	$\frac{\sqrt{n-1}}{2}$	*Yes*	$\frac{n+1}{2}$	$\frac{\sqrt{n-1}}{2}$	*Yes*
$\{132, 123\}$	$\to 2$	$\to 0$	*No*	$\frac{3n}{4}$	$\sim \frac{\sqrt{n}}{4}$	*Yes*
$\{132, 213\}$	$\to \log_2 n$	$\to$ *constant*	*No*	$\frac{n+1}{2}$	$\frac{\sqrt{n-1}}{2}$	*Yes*

Third case: LIS_n on $S_n(\tau^1, \tau^2)$ where $\tau^1 \in S_3$ and $\tau^2 \in S_4$

Thanks to the symmetries, it mainly suffices to study LIS_n on

$$S_n(312, \tau) \text{ where } \tau \in S_k(312)$$

and

$$S_n(213, \tau) \text{ where } \tau \in S_k(213).$$

Note that if $\tau \notin S_k(312)$, then $S_n(312, \tau) = S_n(312)$ for all $n \geq 1$.

For any $\tau \in S_k(312)$, we define **the generating function**

$$F_\tau(x, q) = \sum_{n \geq 0} \sum_{\sigma \in S_n(312, \tau)} x^n q^{\mathrm{LIS}_n(\sigma)}$$

Note that

$$\mathbb{E}^\tau(\mathrm{LIS}_n) = \frac{[x^n] \frac{\partial}{\partial q} F_\tau(x, q) \mid_{q=1}}{[x^n] F_\tau(x, 1)}$$

LIS_n on $S_n(312, \tau)$ where $\tau \in S_k(312)$

To determine $F_\tau(x, q)$ explicitly, we shall introduce some notations.

For any sequence $w = w_1 w_2 \cdots w_m$ of m-distinct integers, we define the corresponding **reduced form** to be the unique permutation $v = v_1 v_2 \cdots v_m$ where $v_i = \ell$ if the w_i is the ℓ-th smallest term in w.

For any sequence w, we define $F_w(x, q)$ to be $F_v(x, q)$ where v is the *reduced form* of w.

Let w^1, w^2 be two sequences of integers, we write

$$w^1 < w^2 \text{ or } w^2 > w^1 \text{ if } w^1_i < w^2_j \text{ for all possible} i, j.$$

The normal form of $\tau \in S_k(312)$

For $\tau = \tau_1 \cdots \tau_k$, τ_i is called a **right-to-left minimum** if $\tau_i < \tau_j$ for all $j > i$.
Let $m_0 = 1 < m_1 < \ldots < m_r$ be the right-to-left minima of τ written from left to right.

Then τ can be represented as

$$\tau = \tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(r)} m_r,$$

where $m_0 < \tau^{(0)} < m_1 < \tau^{(1)} < \cdots < m_r < \tau^{(r)}$, and $\tau^{(j)}$ (may possibly be empty) avoids 312 for each $j = 0, 1, \ldots, r$.

We call this representation the **normal form** of τ.
We define

$$\Theta^{(j)} = \tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(j)} m_j$$

$$\Theta^{<j>} = \text{ the reduced form of } \tau^{(j)} m_j \tau^{(j+1)} m_{j+1} \cdots \tau^{(r)} m_r.$$

LIS_n on $S_n(312, \tau)$ where $\tau \in S_k(312)$

Our main result gives a functional equation for $F_\tau(x, q)$. Assume that τ is written in its normal form: $\tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(r)} m_r$.

Theorem (Mansour - Y. (2018))

If $\tau^{(0)} = \emptyset$, then

$$\begin{aligned} F_\tau(x,q) &= 1 + xq + x(F_\tau(x,q) - 1) + xq(F_{\Theta^{<1>}}(x,q) - 1) \\ &\quad + x\sum_{j=1}^{r}(F_{\Theta^{(j)}}(x,q) - F_{\Theta^{(j-1)}}(x,q))(F_{\Theta^{<j>}}(x,q) - 1); \end{aligned}$$

if $\tau^{(0)} \neq \emptyset$, then $F_\tau(x, q)$

$$\begin{aligned} &= 1 + xq + x(F_{\tau^{(0)}}(x,q) - 1)\delta_{r=0} + x(F_\tau(x,q) - 1)\delta_{r\geq 1} \\ &+ xq(F_\tau(x,q) - 1) + x\sum_{j=2}^{r}(F_{\Theta^{(j)}}(x,q) - F_{\Theta^{(j-1)}}(x,q))(F_{\Theta^{<j>}}(x,q) - 1) \\ &+ x(F_{\Theta^{(1)}}(x,q) - F_{\tau^{(0)}}(x,q))(F_{\Theta^{<1>}}(x,q) - 1)\delta_{r\geq 1} \\ &+ x(F_{\tau^{(0)}}(x,q) - 1)(F_\tau(x,q) - 1) \end{aligned}$$

where $F_\emptyset(x, q) = 0$, and δ_χ denotes 1 if the condition χ holds, and 0 otherwise.

$\mathbb{E}^{\tau}(\mathrm{LIS}_n)$ on $S_n(312, \tau)$ with $\tau \in S_4(312)$

Mansour and Y. (2018)

A corollary of the previous theorem gives us the following:

τ	$\mathbb{E}^{\tau}(\mathrm{LIS}_n)$
1234	$\to 3$
1243,1324	$\sim \frac{n}{2}$
2134,2314 1342	$\sim \frac{n}{\sqrt{5}}$
2143,3214 2431,3241 3421,1432	$\sim \frac{n}{\sqrt{5}}$
2341,432	$\sim \frac{(-5a^2+22a-9)n}{31}$ $a \approx 2.46577\cdots,\ a^3 - 4a^2 + 5a - 3 = 0$

$\mathbb{E}^{\tau}(\text{LIS}_n)$ on $S_n(213,\tau)$ with $\tau \in S_4(213)$

Mansour and Y. (2020)

For any $\tau \in S_k(213)$, we define **the generating function**

$$F_\tau(x,q) = \sum_{n\geq 0} \sum_{\sigma \in S_n(213,\tau)} x^n q^{\text{LIS}_n(\sigma)}$$

Our main result gives a functional equation for $F_\tau(x,q)$ and one of its corollaries leads to the following:

τ	$\mathbb{E}^{\tau}(\text{LIS}_n)$
1234	~ 3
1243	$\sim \log_{\frac{3+\sqrt{5}}{2}} n$
1342	$\sim \log_{\frac{\sqrt{5}+1}{2}} n$
3412	$\sim \frac{\sqrt{5}}{5} n$
4123, 4312	$\sim \frac{5-\sqrt{5}}{10} n$
4231	$\sim \frac{3}{8} n$
4321	$\sim \frac{11}{15} n$

LIS_n on involutions $\mathbf{Inv}_\mathrm{n} = \{\sigma \in S_n : \sigma^{-1} = \sigma\}$

Theorem (Baik-Rains (2001))

Consider $\mathbf{Inv}_\mathrm{n}$ *with the uniform probability distribution. Then*

$$\lim_{n\to\infty} Pr\left(\frac{\text{LIS}_n - 2\sqrt{n}}{n^{1/6}} \le t\right) = F_{GOE}(t) \quad \textit{for all} \quad t \in \mathbb{R},$$

where F_{GOE} *is the Tracy-Widom GOE distribution function.*

- T. Mansour and G. Yıldırım studied the average length of LIS_n on $\mathbf{Inv}_\mathrm{n}(\tau^1, \tau^2)$ where $\tau^1, \tau^2 \in S_3$. (2019)

- T. Mansour, R. Rastegar, A. Roitershtein and G. Yıldırım studied the average length of LIS_n on $\mathbf{Inv}_\mathrm{n}(3412)$ and on $\mathbf{Inv}_\mathrm{n}(3412, \tau)$, the set of involutions of length n avoiding 3412 and another pattern. (2020)

LIS_n on words over the alphabet $[k] = \{1, 2, \cdots, k\}$

A **word of length** n over the alphabet $[k]$ is a map $\omega : [n] \to [k]$.

Let W_n^k denote the set of all words of length n over $[k]$.

For $\omega \in W_n^k$, we say that $\omega_{i_1}\omega_{i_2}\cdots\omega_{i_l}$ is a **weakly increasing subsequence of length** l in ω if

$$i_1 < i_2 < \cdots < i_l \text{ and } \omega_{i_1} \le \omega_{i_2} \le \cdots \le \omega_{i_l}.$$

Let

$$\mathrm{LIS}_n^{wk}(\omega) = \text{the length of the } \textbf{l.w.i.} \text{ subsequence in } \omega.$$

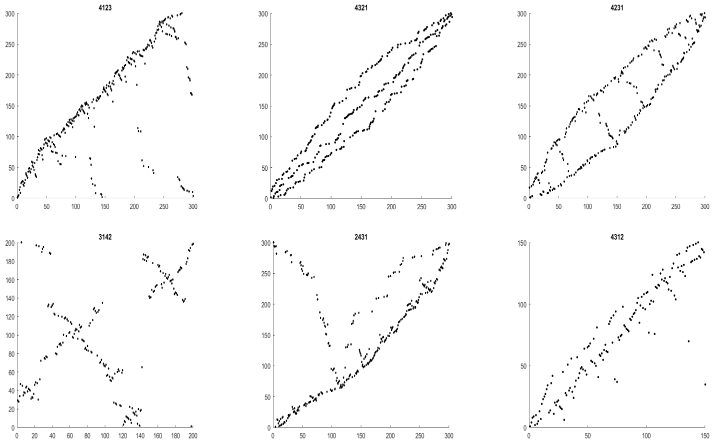
Theorem (Tracy-Widom (2001))

Consider W_n^k with the uniform probability distribution. Then

$$\lim_{n\to\infty} Pr\left(\frac{\mathrm{LIS}_n^{wk} - n/k}{\sqrt{\frac{2n}{k}}} \le t\right) = F_o(t) \quad \textit{for all} \quad t \in \mathbb{R},$$

where F_o distribution function for the largest eigenvalue of a $k \times k$ GUE matrix conditioned to have trace zero.

Thank you!



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