Longest increasing subsequences in pattern-avoiding permutations

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Longest increasing subsequences in permutations

The longest increasing subsequence problem for uniformly random permutations has a long and interesting history. The research around this problem, also called Ulam's problem, has made surprising connections among different fields such as combinatorics, probability, statistical physics and computer science.

We use one-line notation for permutations \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \).

\( S_n \) denotes the set of all permutations of length \( n \).

For \( \sigma \in S_n \), we say that \( \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_l} \) is an increasing subsequence of length \( l \) in \( \sigma \) if \( i_1 < i_2 < \cdots < i_l \) and \( \sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_l} \).

Let

\[
\text{LIS}_n(\sigma) = \text{the length of the longest increasing subsequence in } \sigma.
\]

\[
\text{LDS}_n(\sigma) = \text{the length of the longest decreasing subsequence in } \sigma.
\]
A brief history of Ulam’s problem

Erdös-Szekeres lemma (1935):

For every $\sigma \in S_n$, $\text{LIS}_n(\sigma) \text{LDS}_n(\sigma) \geq n$

In 1961, on the basis of some simulations, Ulam conjectured that

$$c_o := \lim_{n \to \infty} \frac{\mathbb{E}(\text{LIS}_n)}{\sqrt{n}}$$

determines the limit.

Hammersley (1972) proved the existence of the limit.

Logan and Shepp (1977), Vershik and Kerov (1977) determined the constant: $c_o = 2$. 
A real breakthrough was achieved by Baik, Deift and Johansson in 1999 by completely determining the asymptotic distribution of LIS$_n$.

**Theorem (Baik-Deift-Johanson (1999))**

Consider $S_n$ with the uniform probability distribution. Then

$$\lim_{n \to \infty} \Pr \left( \frac{\text{LIS}_n - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F_{\text{GUE}}(t) \quad \text{for all} \quad t \in \mathbb{R},$$

where $F_{\text{GUE}}$ is the Tracy-Widom GUE distribution function.

The Tracy-Widom GUE is the largest eigenvalue distribution for the Gaussian Unitary Ensemble from random matrix theory.
An interacting particle process on $[0, 1]$

- Initially there are zero particles in the system.

- At each step, a particle appears at a uniform random point $u$ in the interval $[0, 1]$; simultaneously the nearest particle (if any) to the right of $u$ disappears.

Let $\mathcal{P}_n = \text{the number of particles in the system after } n \text{ steps.}$

Random variables $\text{LIS}_n$ and $\mathcal{P}_n$ have the same probability distribution.

This model also gives a very efficient algorithm to simulate $\text{LIS}_n$ on $S_n$ under the uniform probability distribution.
LIS$_n$ on pattern-avoiding permutations

Let $S_n(\tau) =$ the set of $\tau$-avoiding permutations of length $n$.

some bijections on $S_n$: complement, reverse, inverse

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$.

- the complement of $\sigma$ is $\sigma^c = \sigma_1^c\sigma_2^c\cdots\sigma_n^c$ where $\sigma_i^c = n+1-\sigma_i$ for $i \in [n]$.
- the reverse of $\sigma$ is defined to be $\sigma^r = \sigma_n\sigma_{n-1}\cdots\sigma_1$.
- $\tau^{-1}$ is the inverse permutation of $\tau$.

We have

$$|S_n(\tau)| = |S_n(\tau^c)| = |S_n(\tau^r)| = |S_n(\tau^{-1})|.$$

Note that

$$\text{LIS}_n(\sigma) = \text{LIS}_n(\sigma^{rc}) = \text{LIS}_n(\sigma^{-1}) = \text{LIS}_n((\sigma^{rc})^{-1})$$

and

$$\text{LIS}_n(\sigma) = \text{LDS}_n(\sigma^r)$$
First case: $\text{LIS}_n$ on $S_n(\tau)$ where $\tau \in S_3$

**Theorem (Deutsch-Hildebrand-Wilf (2003))**

a- *In the case* $\tau = 231$, *for all* $t \in \mathbb{R}$

$$\lim_{n \to \infty} P_{n}^{231} \left( \frac{\text{LIS}_n - (n + 1)/2}{\sqrt{n}/2} \leq t \right) = \Phi(t), \text{ standard normal}$$

b- *In the case* $\tau = 132$,

$$\lim_{n \to \infty} P_{n}^{132} \left( \frac{\text{LIS}_n - \sqrt{\pi n}}{\sqrt{n}} \leq t \right) = \sum_{s=-\infty}^{\infty} (1 - 2s^2(t + \sqrt{\pi})^2)e^{-(t + \sqrt{\pi})^2s^2}$$

*for* $t > -\sqrt{\pi}$.

c- *In the case* $\tau = 321$, *for all* $t \in \mathbb{R}$

$$\lim_{n \to \infty} P_{n}^{321} \left( \frac{\text{LIS}_n - n/2}{\sqrt{n}} \leq t \right) = \frac{2}{\sqrt{\pi}} \int_{0}^{4t^2} u^{1/2} e^{-u} du$$
Second case: LIS\(_n\) on \(S_n(\tau^1, \tau^2)\) where \(\tau^1, \tau^2 \in S_3\)

**Theorem (Madras - Y. (2017))**

Consider \(S_n(\tau^1, \tau^2)\) with uniform probability distribution where \(\tau^1, \tau^2 \in S_3\). Then we have:

<table>
<thead>
<tr>
<th>{\tau_1, \tau_2}</th>
<th>LIS(_n)</th>
<th>LDS(_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>SD</td>
<td>asymptotically normal?</td>
</tr>
<tr>
<td>{132, 321}</td>
<td>(\sim \frac{5n}{6})</td>
<td>(\sim \frac{5n}{6\sqrt{2}})</td>
</tr>
<tr>
<td>{132, 231}</td>
<td>(\frac{n+1}{2})</td>
<td>(\frac{\sqrt{n-1}}{2})</td>
</tr>
<tr>
<td>{132, 123}</td>
<td>(\rightarrow 2)</td>
<td>(\rightarrow 0)</td>
</tr>
<tr>
<td>{132, 213}</td>
<td>(\rightarrow \log_2 n)</td>
<td>(\rightarrow \text{constant})</td>
</tr>
</tbody>
</table>
Third case: \( \text{LIS}_n \) on \( S_n(\tau^1, \tau^2) \) where \( \tau^1 \in S_3 \) and \( \tau^2 \in S_4 \)

Thanks to the symmetries, it mainly suffices to study \( \text{LIS}_n \) on

\[
S_n(312, \tau) \text{ where } \tau \in S_k(312)
\]

and

\[
S_n(213, \tau) \text{ where } \tau \in S_k(213).
\]

Note that if \( \tau \not\in S_k(312) \), then \( S_n(312, \tau) = S_n(312) \) for all \( n \geq 1 \).

For any \( \tau \in S_k(312) \), we define the generating function

\[
F_\tau(x, q) = \sum_{n \geq 0} \sum_{\sigma \in S_n(312, \tau)} x^n q^{\text{LIS}_n(\sigma)}
\]

Note that

\[
\mathbb{E}^\tau(\text{LIS}_n) = [x^n] \frac{\partial}{\partial q} F_\tau(x, q) \bigg|_{q=1} \frac{[x^n] F_\tau(x, 1)}{[x^n] F_\tau(x, 1)}
\]
To determine $F_\tau(x, q)$ explicitly, we shall introduce some notations.

For any sequence $w = w_1 w_2 \cdots w_m$ of $m$-distinct integers, we define the corresponding **reduced form** to be the unique permutation $v = v_1 v_2 \cdots v_m$ where $v_i = \ell$ if the $w_i$ is the $\ell$-th smallest term in $w$.

For any sequence $w$, we define $F_w(x, q)$ to be $F_v(x, q)$ where $v$ is the reduced form of $w$.

Let $w^1, w^2$ be two sequences of integers, we write

$$w^1 < w^2 \text{ or } w^2 > w^1 \text{ if } w^1_i < w^2_j \text{ for all possible } i, j.$$
The normal form of \( \tau \in S_k(312) \)

For \( \tau = \tau_1 \cdots \tau_k \), \( \tau_i \) is called a **right-to-left minimum** if \( \tau_i < \tau_j \) for all \( j > i \).

Let \( m_0 = 1 < m_1 < \ldots < m_r \) be the right-to-left minima of \( \tau \) written from left to right.

Then \( \tau \) can be represented as

\[
\tau = \tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(r)} m_r,
\]

where \( m_0 < \tau^{(0)} < m_1 < \tau^{(1)} < \cdots m_r < \tau^{(r)} \), and \( \tau^{(j)} \) (may possibly be empty) avoids 312 for each \( j = 0, 1, \ldots, r \).

We call this representation the **normal form** of \( \tau \).

We define

\[
\Theta^{(j)} = \tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(j)} m_j
\]

\[
\Theta^{<j>} = \text{the reduced form of } \tau^{(j)} m_j \tau^{(j+1)} m_{j+1} \cdots \tau^{(r)} m_r.
\]
Our main result gives a functional equation for $F_\tau(x, q)$. Assume that $\tau$ is written in its normal form: $\tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(r)} m_r$.

**Theorem (Mansour - Y. (2018))**

If $\tau^{(0)} = \emptyset$, then

$$F_\tau(x, q) = 1 + x q + x (F_\tau(x, q) - 1) + x q (F_{\Theta<1>}(x, q) - 1)$$

$$+ x \sum_{j=1}^{r} (F_{\Theta(j)}(x, q) - F_{\Theta(j-1)}(x, q))(F_{\Theta<j>}(x, q) - 1);$$

if $\tau^{(0)} \neq \emptyset$, then $F_\tau(x, q)$

$$= 1 + x q + x (F_{\tau^{(0)}}(x, q) - 1)\delta_{r=0} + x (F_\tau(x, q) - 1)\delta_{r\geq1}$$

$$+ x q (F_\tau(x, q) - 1) + x \sum_{j=2}^{r} (F_{\Theta(j)}(x, q) - F_{\Theta(j-1)}(x, q))(F_{\Theta<j>}(x, q) - 1)$$

$$+ x (F_{\Theta^{(1)}}(x, q) - F_{\tau^{(0)}}(x, q))(F_{\Theta<1>}(x, q) - 1)\delta_{r\geq1}$$

$$+ x (F_{\tau^{(0)}}(x, q) - 1)(F_\tau(x, q) - 1)$$

where $F_\emptyset(x, q) = 0$, and $\delta_{\chi}$ denotes 1 if the condition $\chi$ holds, and 0 otherwise.
A corollary of the previous theorem gives us the following:

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( \mathbb{E}^{\tau}(\text{LIS}_n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1234</td>
<td>( \rightarrow 3 )</td>
</tr>
<tr>
<td>1243,1324</td>
<td>( \sim \frac{n}{2} )</td>
</tr>
<tr>
<td>2134,2314</td>
<td>( \sim \frac{n}{\sqrt{5}} )</td>
</tr>
<tr>
<td>1342</td>
<td></td>
</tr>
<tr>
<td>2143,3214</td>
<td>( \sim \frac{n}{\sqrt{5}} )</td>
</tr>
<tr>
<td>2431,3241</td>
<td></td>
</tr>
<tr>
<td>3421,1432</td>
<td></td>
</tr>
<tr>
<td>2341,432</td>
<td>( \sim \frac{(-5a^2+22a-9)n}{31} )</td>
</tr>
</tbody>
</table>

\( a \approx 2.46577 \ldots \), \( a^3 - 4a^2 + 5a - 3 = 0 \)
For any $\tau \in S_k(213)$, we define the generating function

$$F_\tau(x, q) = \sum_{n \geq 0} \sum_{\sigma \in S_n(213, \tau)} x^n q^{LIS_n(\sigma)}$$

Our main result gives a functional equation for $F_\tau(x, q)$ and one of its corollaries leads to the following:

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\mathbb{E}^\tau(LIS_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1234</td>
<td>$\sim 3$</td>
</tr>
<tr>
<td>1243</td>
<td>$\sim \log_{\frac{3+\sqrt{5}}{2}} n$</td>
</tr>
<tr>
<td>1342</td>
<td>$\sim \log_{\frac{\sqrt{5}+1}{2}} n$</td>
</tr>
<tr>
<td>3412</td>
<td>$\sim \frac{\sqrt{5}}{5} n$</td>
</tr>
<tr>
<td>4123, 4312</td>
<td>$\sim \frac{5-\sqrt{5}}{10} n$</td>
</tr>
<tr>
<td>4231</td>
<td>$\sim \frac{3}{8} n$</td>
</tr>
<tr>
<td>4321</td>
<td>$\sim \frac{11}{15} n$</td>
</tr>
</tbody>
</table>
LIS\textsubscript{n} on involutions \textbf{Inv\textsubscript{n}} = \{\sigma \in S\textsubscript{n} : \sigma^{-1} = \sigma\}

Theorem (Baik-Rains (2001))

Consider \textbf{Inv\textsubscript{n}} with the uniform probability distribution. Then

\[ \lim_{n \to \infty} Pr \left( \frac{\text{LIS}_n - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F_{GOE}(t) \text{ for all } t \in \mathbb{R}, \]

where \(F_{GOE}\) is the Tracy-Widom GOE distribution function.

- T. Mansour and G. Yıldırım studied the average length of LIS\textsubscript{n} on \textbf{Inv\textsubscript{n}}(\tau^1, \tau^2) where \(\tau^1, \tau^2 \in S_3\). (2019)

- T. Mansour, R. Rastegar, A. Roitershtein and G. Yıldırım studied the average length of LIS\textsubscript{n} on \textbf{Inv\textsubscript{n}}(3412) and on \textbf{Inv\textsubscript{n}}(3412, \tau), the set of involutions of length \(n\) avoiding 3412 and another pattern. (2020)
LIS\(_n\) on words over the alphabet \([k] = \{1, 2, \cdots, k\}\)

A word of length \(n\) over the alphabet \([k]\) is a map \(\omega : [n] \to [k]\).

Let \(W_n^k\) denote the set of all words of length \(n\) over \([k]\).

For \(\omega \in W_n^k\), we say that \(\omega_{i_1}\omega_{i_2}\cdots\omega_{i_l}\) is a weakly increasing subsequence of length \(l\) in \(\omega\) if

\[i_1 < i_2 < \cdots < i_l \text{ and } \omega_{i_1} \leq \omega_{i_2} \leq \cdots \leq \omega_{i_l}.\]

Let

\[\text{LIS}^w_n(\omega) = \text{the length of the l.w.i. subsequence in } \omega.\]

**Theorem (Tracy-Widom (2001))**

Consider \(W_n^k\) with the uniform probability distribution. Then

\[
\lim_{n \to \infty} \Pr \left( \frac{\text{LIS}^w_n - n/k}{\sqrt{2n/k}} \leq t \right) = F_o(t) \quad \text{for all } \ t \in \mathbb{R},
\]

where \(F_o\) distribution function for the largest eigenvalue of a \(k \times k\) GUE matrix conditioned to have trace zero.
- G. Yıldırım was partially supported by Tubitak-Grant no: 118C029.
References


T. Mansour and G. Yildirim. *The mean length of the longest increasing subsequences in random permutations avoiding one pattern of length three and another pattern*. (preprint)


T. Mansour, R. Rastegar, A. Roitershtein and G. Yildirim. *The longest increasing subsequence in involutions avoiding 3412 and another pattern*. (preprint)