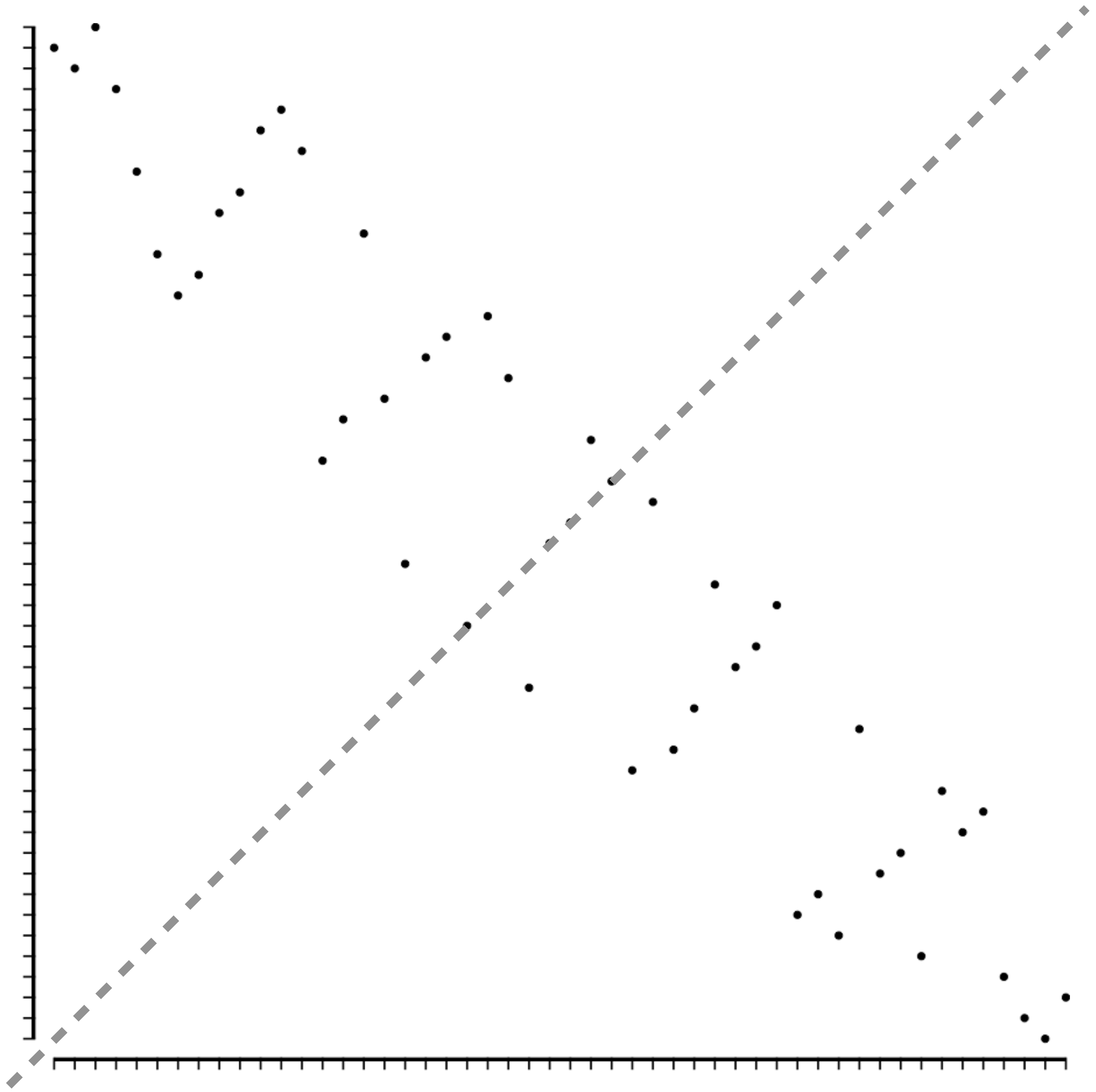
Pattern-Avoiding Involution Classes and their Growth Rates

Jay Pantone

joint work with [Miklós Bóna, Cheyne Homberger, and Vince Vatter] and [John Engbers and Christopher Stocker]

Involutions

- An *involution* is a permutation that is its own group-theoretic inverse, or equivalently, one whose plot has symmetry over the line $y = x$.



Involution Class Enumeration

- We use $Av^I(B)$ to denote the set of involutions that avoid all the patterns in B.

- $Av_n^I(B)$ denotes the involutions in $Av^I(B)$ of size n.

- The *exponential growth rate* of an involution class is

$$\lim_{n\to\infty} \sqrt[n]{|Av_n^I(B)|}$$

- Does it always exist?

Involution Class Enumeration

Bóna, Homberger, P., Vatter (2015)

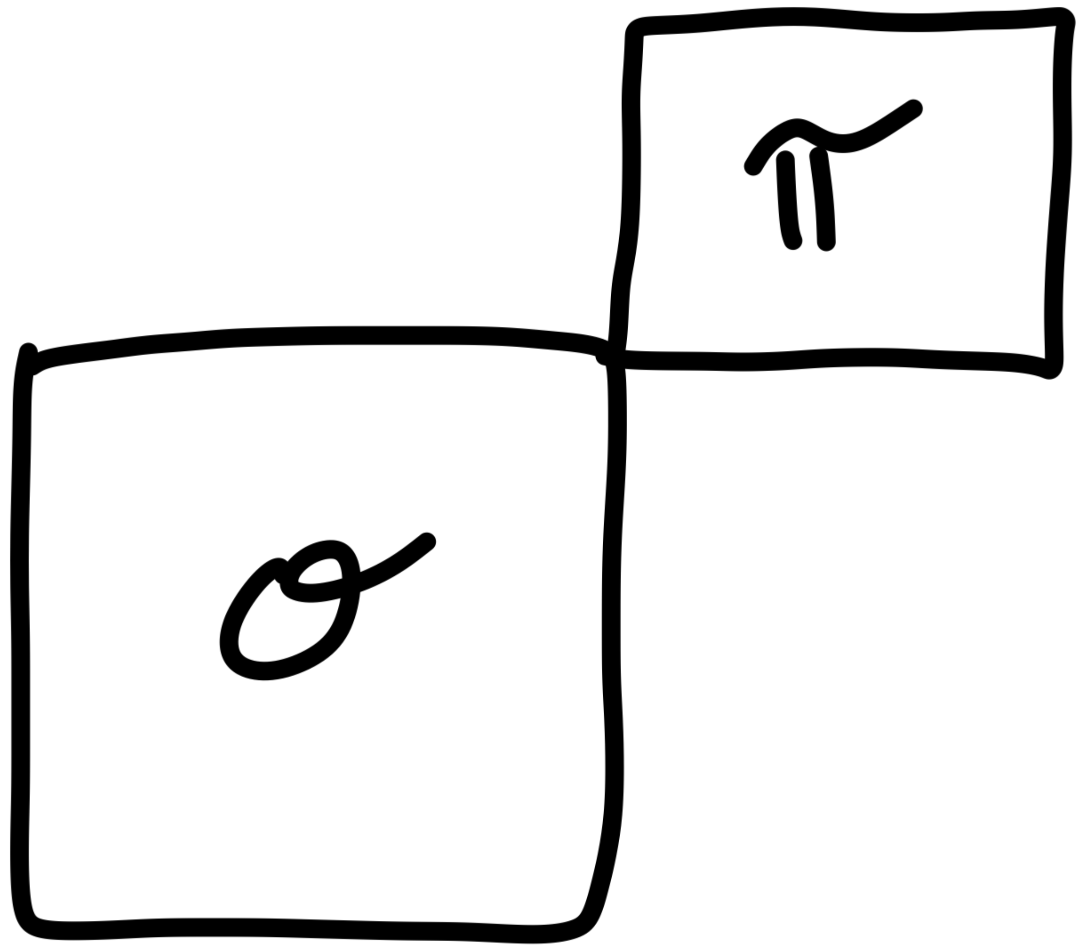
	2431	**2341**	**1342**	**1234**	**1324**	**3421**	**4231**	**2413**
$\lvert \mathrm{Av}^I_{12}(\beta) \rvert$	16238	18477	18322	15511	15272	22878	16716	27246
$\lvert \mathrm{Av}^I_{13}(\beta) \rvert$	40914	46825	47560	41835	40758	60794	46246	77132
$\lvert \mathrm{Av}^I_{14}(\beta) \rvert$	103954	118917	124358	113634	112280	161668	128414	221336
$\lvert \mathrm{Av}^I_{15}(\beta) \rvert$	262298	301734	323708	310572	304471	429752	361493	635078
$\lvert \mathrm{Av}^I_{16}(\beta) \rvert$	665478	766525	846766	853467	852164	1142758	1020506	1839000
$\lvert \mathrm{Av}^I_{17}(\beta) \rvert$	1680726	1946293	2208032	2356779	2341980	3038173	2913060	5331274
$\lvert \mathrm{Av}^I_{18}(\beta) \rvert$	4260262	4944614	5777330	6536382	6640755	8078606	8335405	15555586
$\lvert \mathrm{Av}^I_{19}(\beta) \rvert$	10766470	12557685	15082372	18199284	18460066	21479469	24067930	45465412
$\lvert \mathrm{Av}^I_{20}(\beta) \rvert$	27274444	31900554	39469786	50852019	52915999	57113888	69646035	133517130
growth rate	?	≈ 2.54 Section 6	≈ 2.62 Section 5	3 Regev [26]	> 3.13, < 4.84 Sections 2 & 7	?	?	≈ 3.15 BHV [13]
OEIS	*A230551*	*A230552*	*A230553*	*A001006*	*A230554*	*A230555*	*A230556*	*A121704*

Growth Rates of Principal Classes

Arratia (1999)

- For a single pattern β, the (standard, non-involution) class $Av(\beta)$ has a growth rate.
- Proof: if β is sum-indecomposable, then for all $\sigma, \pi \in Av(\beta)$ we have

$$\sigma \oplus \pi \in Av(\beta)$$



- The map $f : Av_m(\beta) \times Av_n(\beta) \longrightarrow Av_{m+n}(\beta)$ defined by $f(\sigma, \pi) = \sigma \oplus \pi$ is injective.
- So, $|Av_m(\beta)||Av_n(\beta)| \leq |Av_{m+n}(\beta)|$.
- By Fekete's Lemma, $\lim_{n\to\infty} \sqrt[n]{|Av_n(\beta)|}$ exists.
- If β is skew-indecomposable, a symmetric argument applies.

Growth Rates of Involution Classes

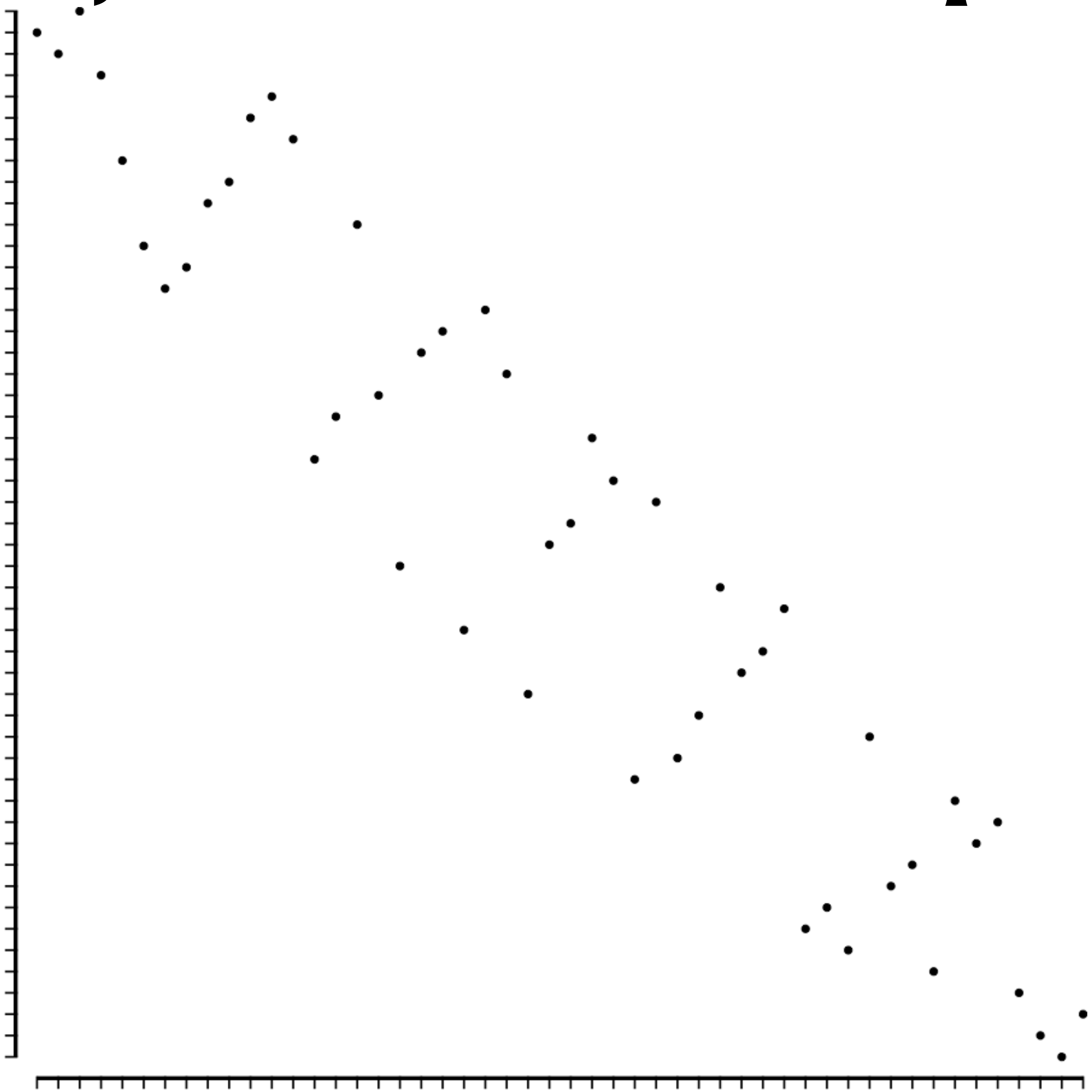
- For an involution class $Av^I(\beta)$, this only works if β is sum-indecomposable!
  - If $\sigma, \pi \in Av^I(\beta)$, then $\sigma \oplus \pi \in Av^I(\beta)$.
  - If β is skew-indecomposable, you need to use $\sigma \ominus \pi$, but this is probably not an involution!

- So, $Av^I(4231)$ has a growth rate. Does $Av^I(1324)$?
  - **Yes**, and so does $Av^I(1 \oplus \pi \oplus 1)$ for any π. — Engbers, P. , Stocker (2020+)

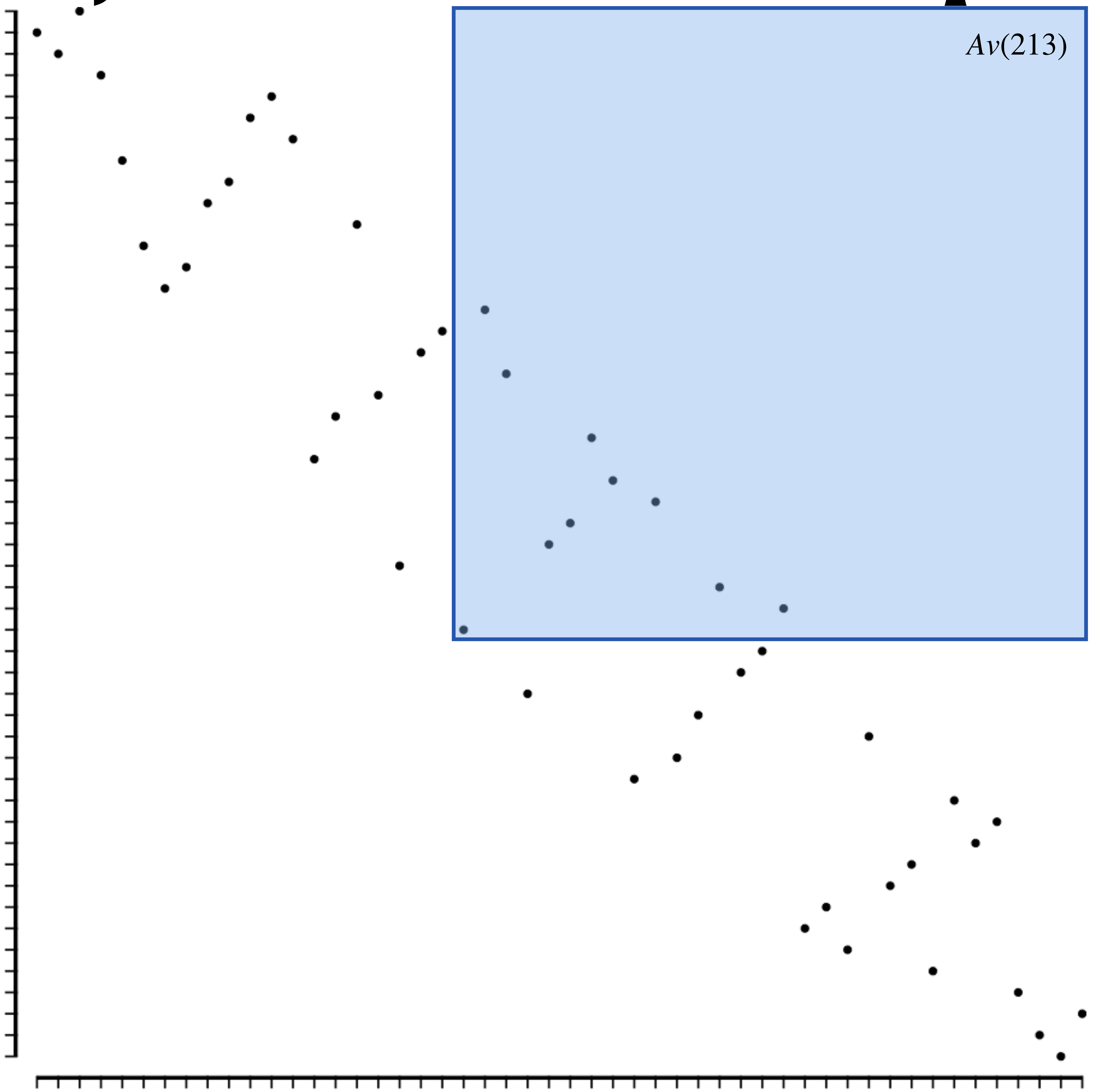
1324-Avoiding Involutions

- We need a much more complicated way of combining two 1324-avoiding involutions in a (mostly) reversible way.
- Strategy:
    - Decompose σ, π each into staircases
    - Decompose them further, splitting each stair into three blocks.
    - Weave the two staircases together in a complicated, reversible way.

Primary Staircase Decomposition



Primary Staircase Decomposition



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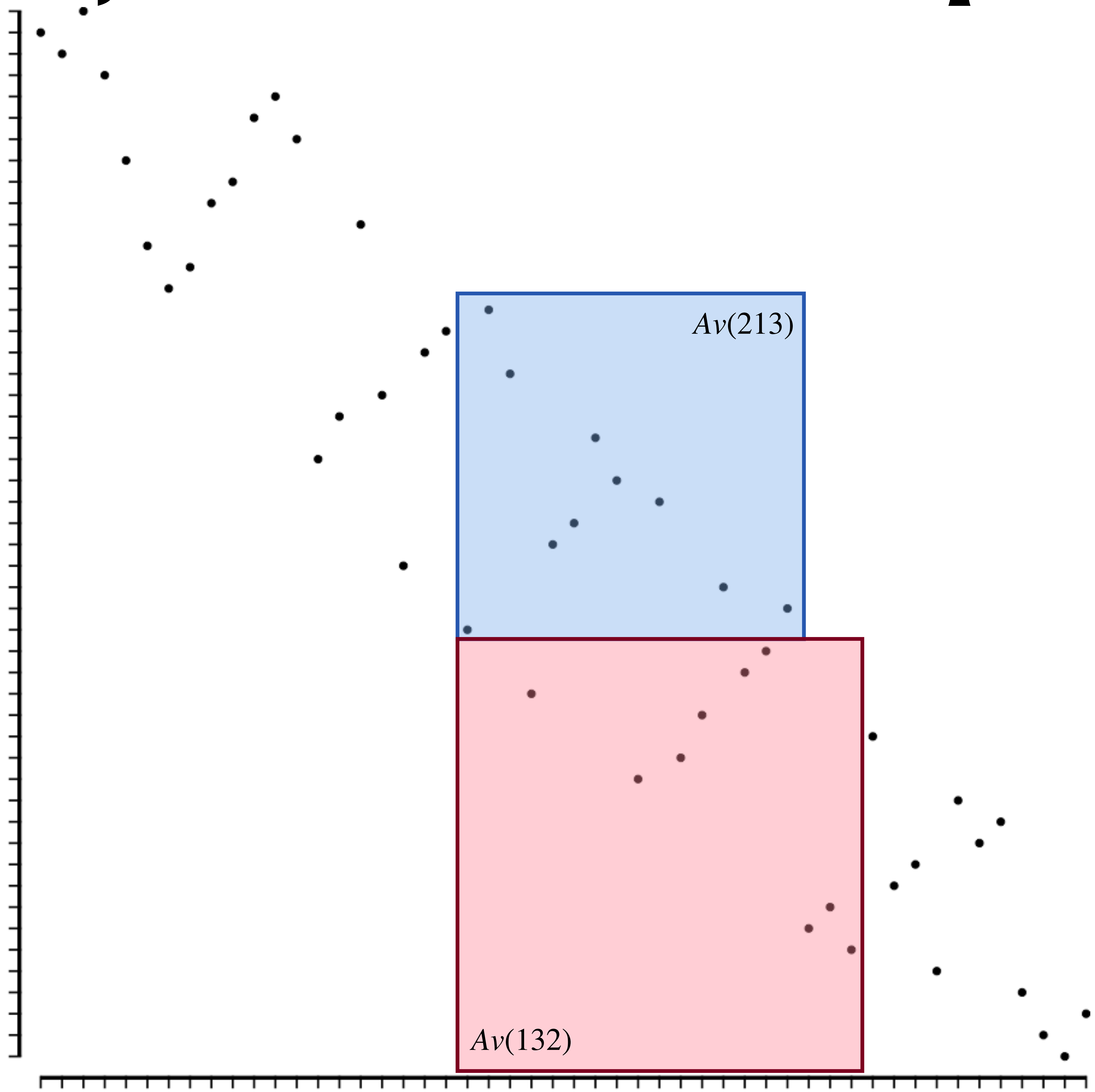
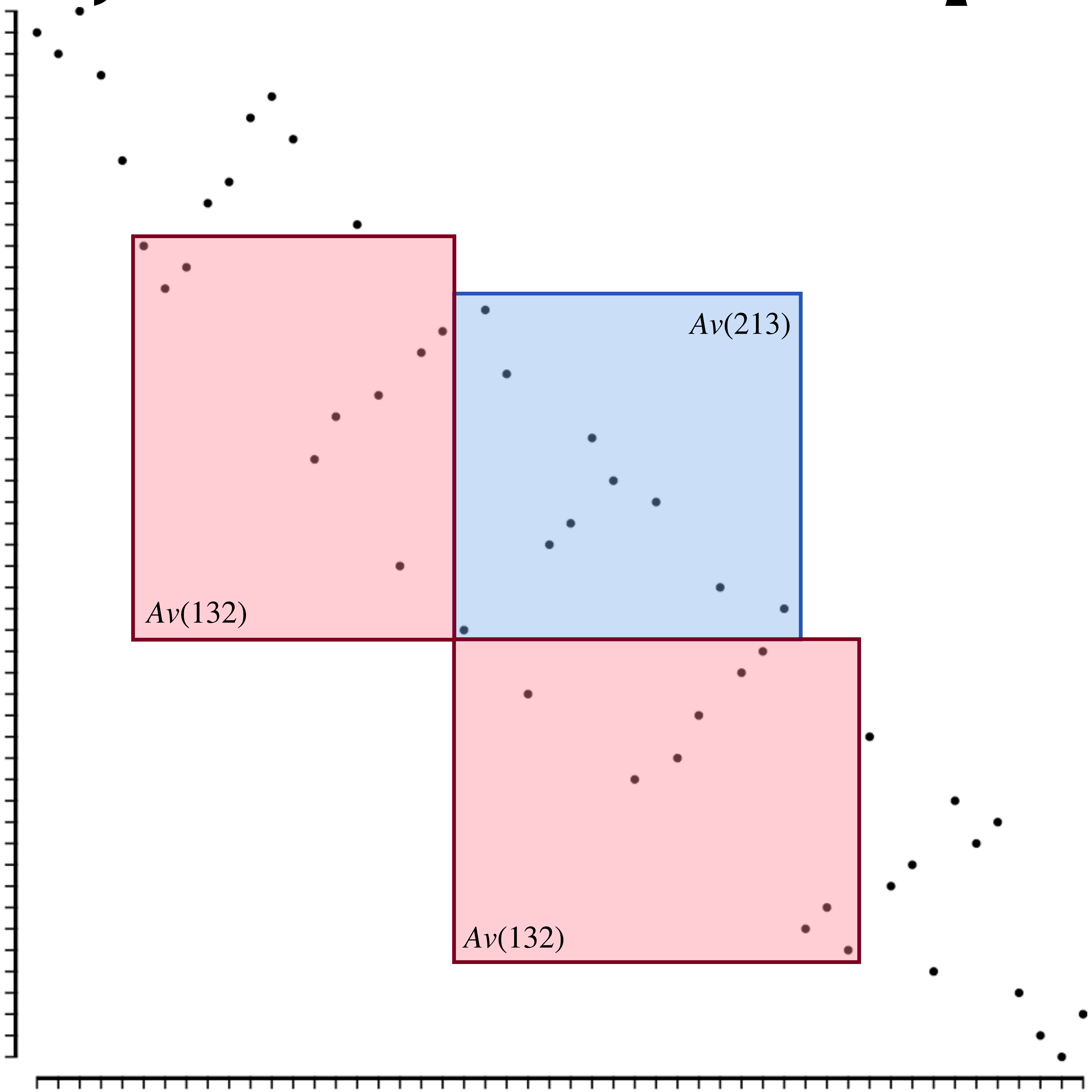


Primary Staircase Decomposition



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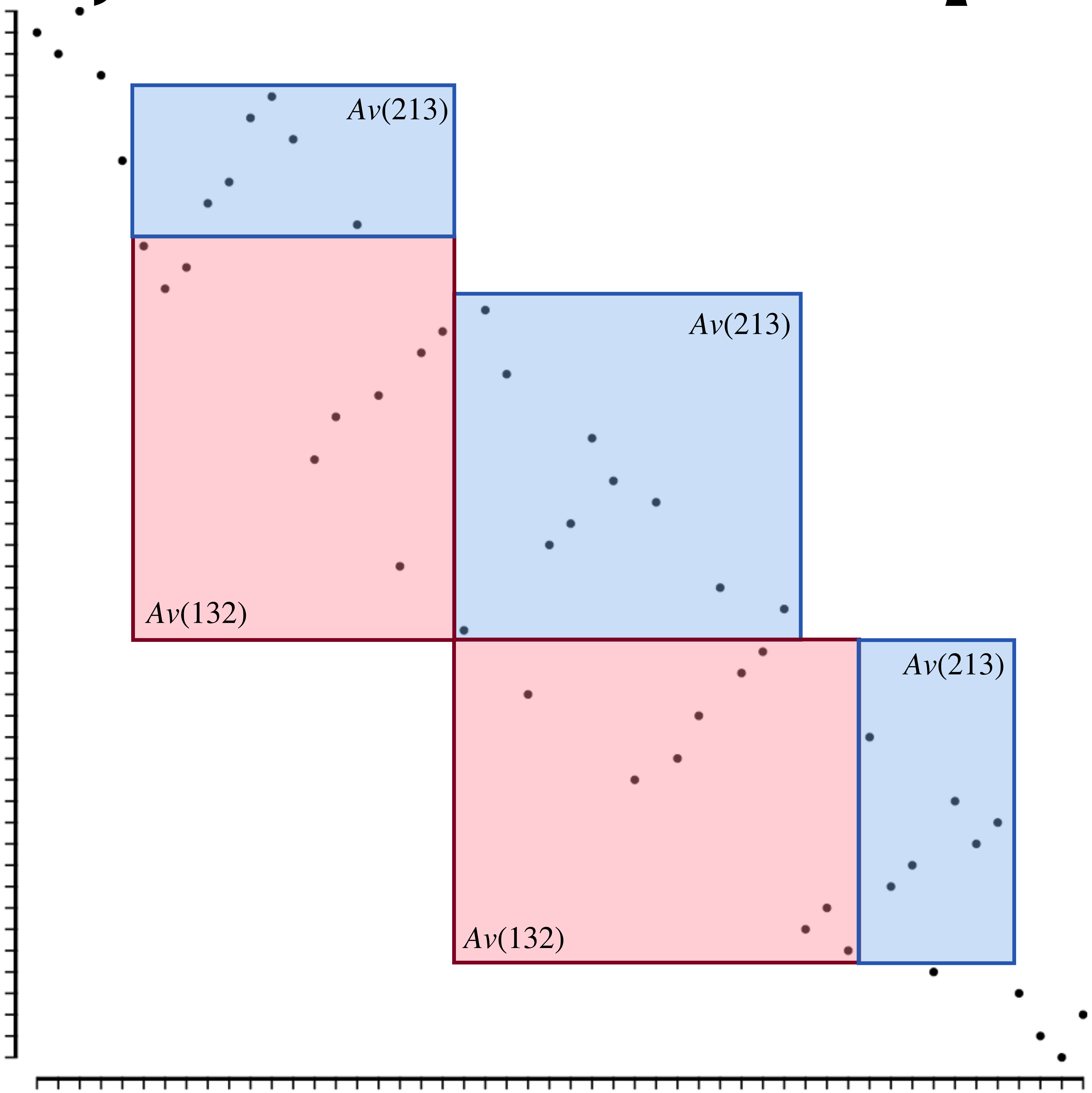

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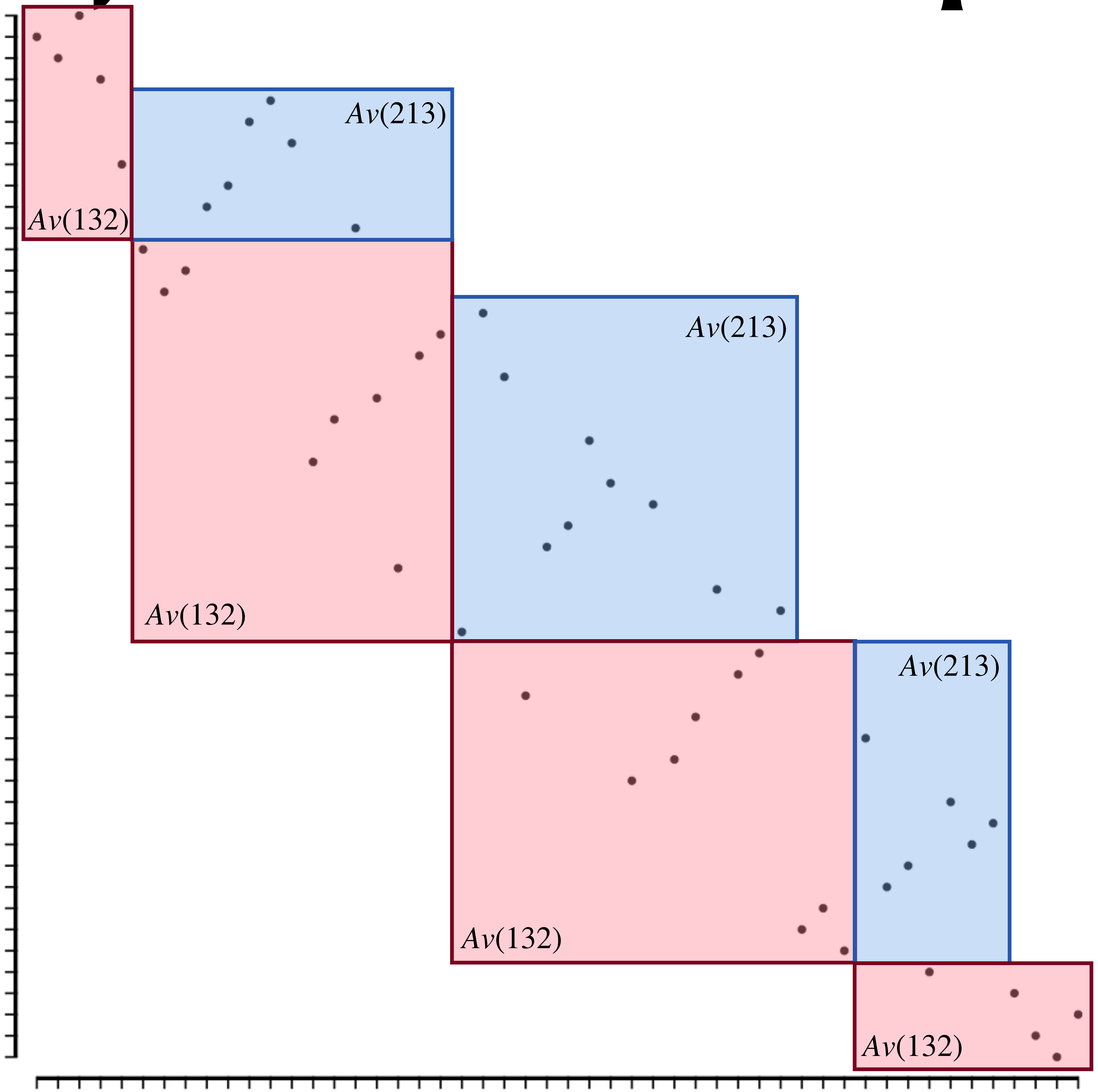
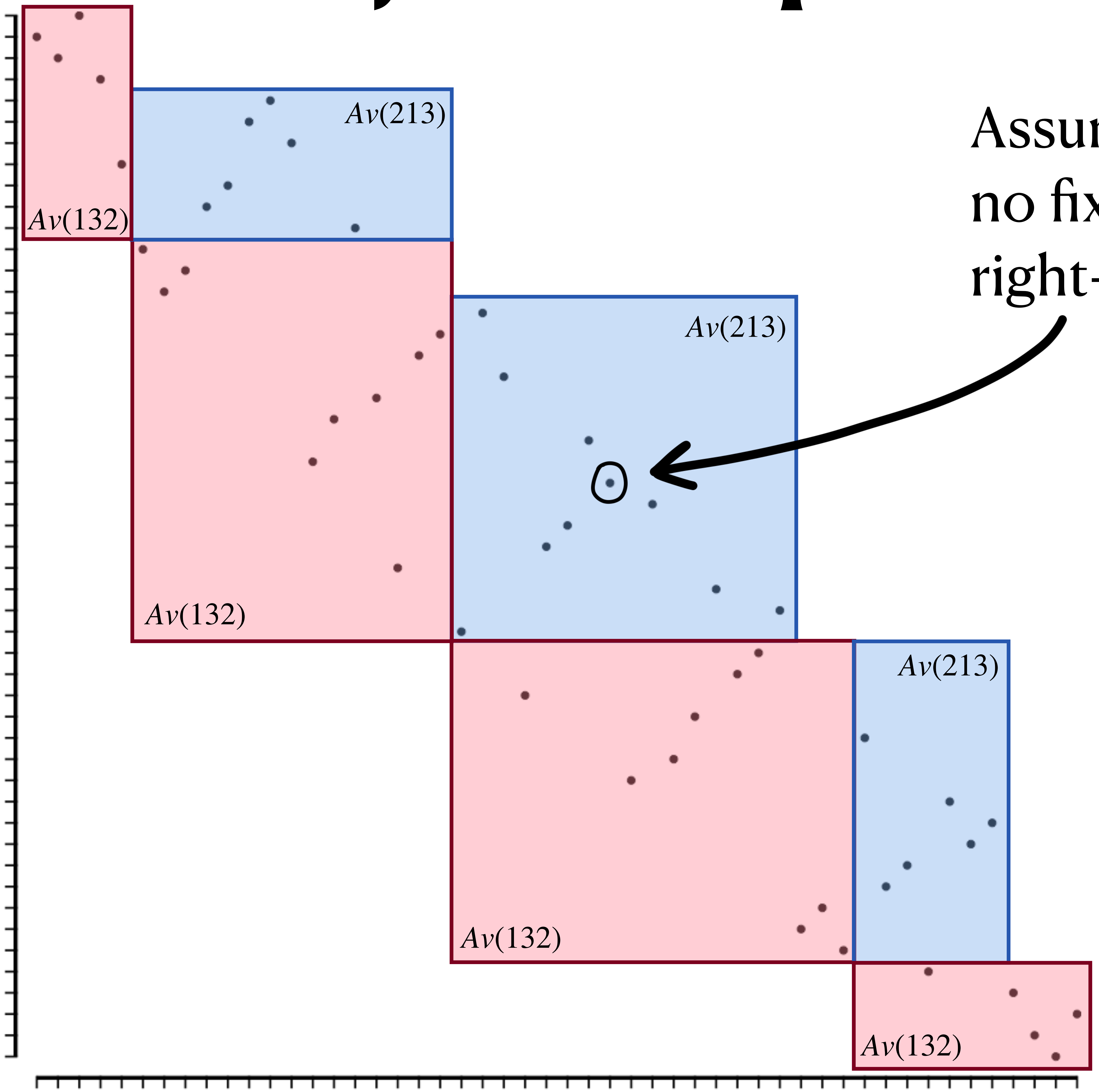
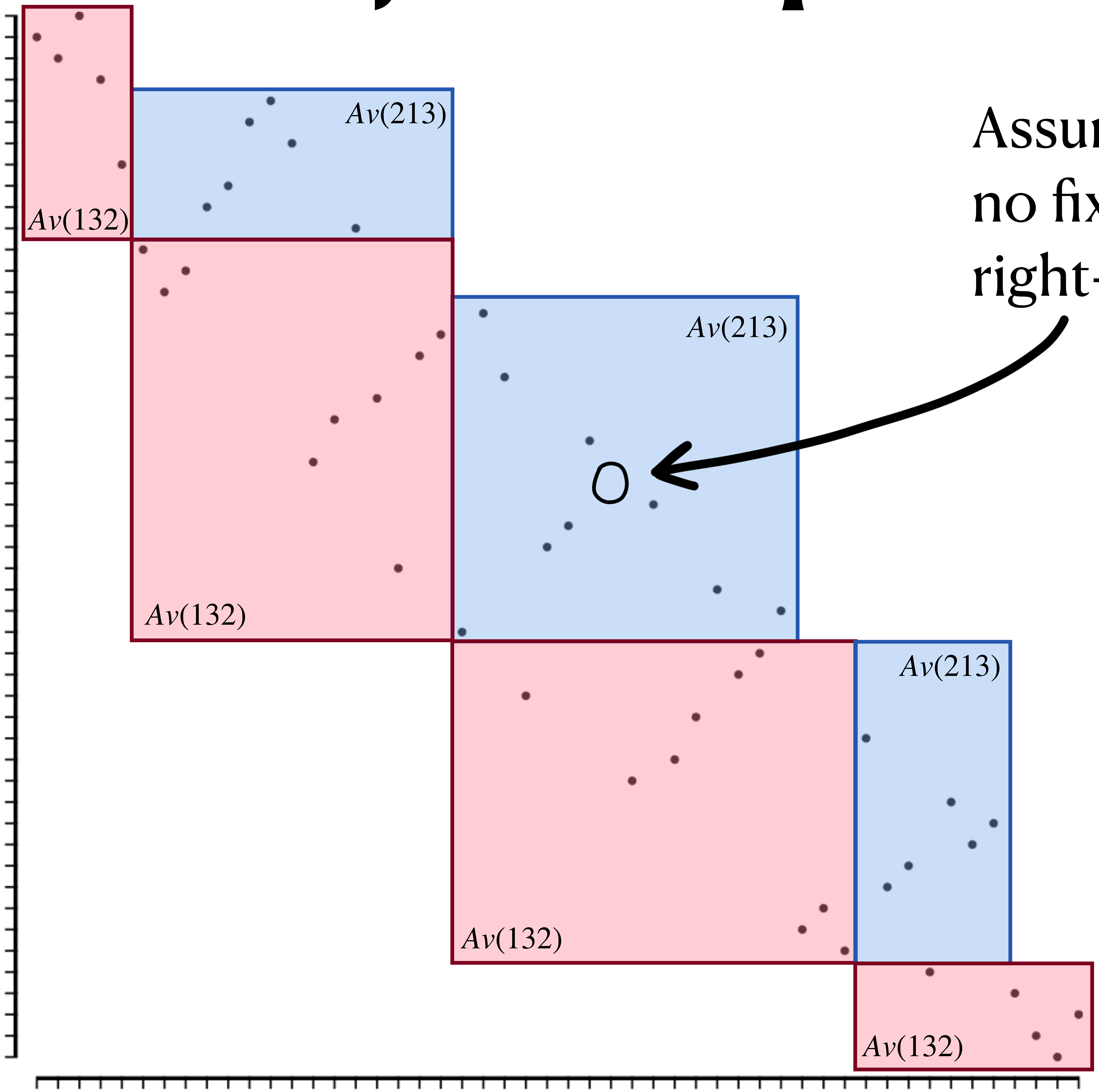


Secondary Decomposition



Assume the involution has no fixed point that is a right-to-left maximum.

Secondary Decomposition



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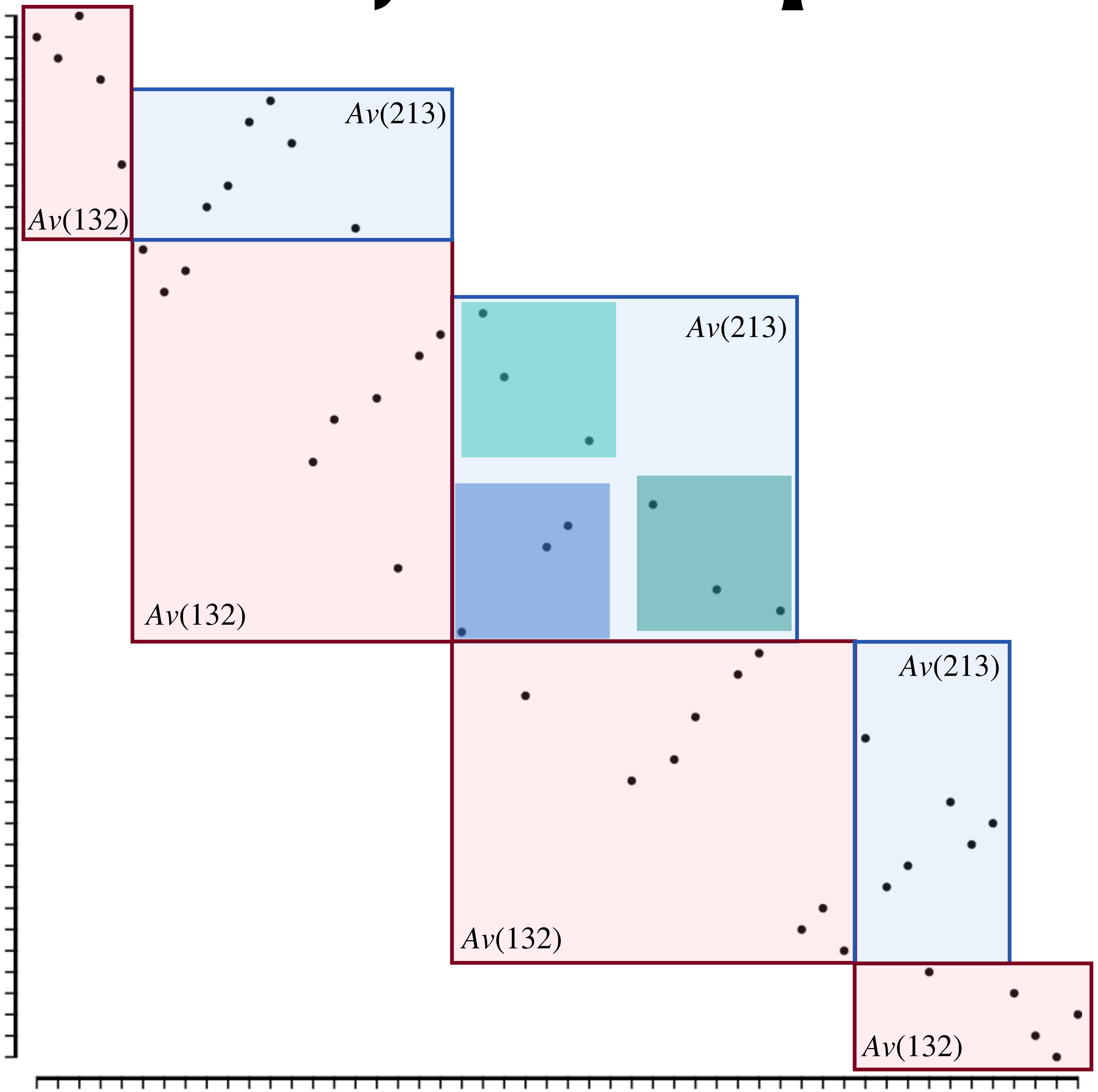

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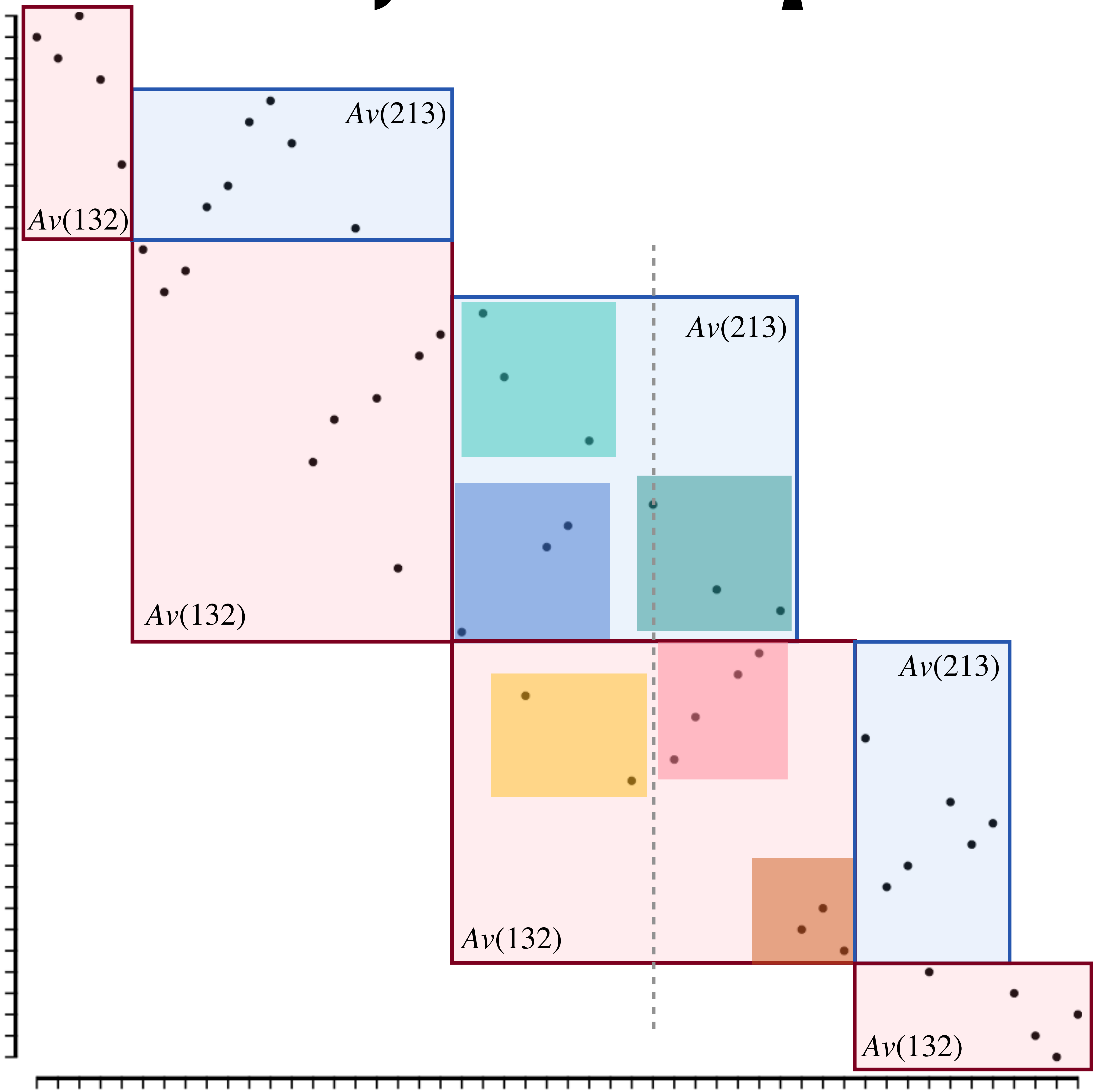

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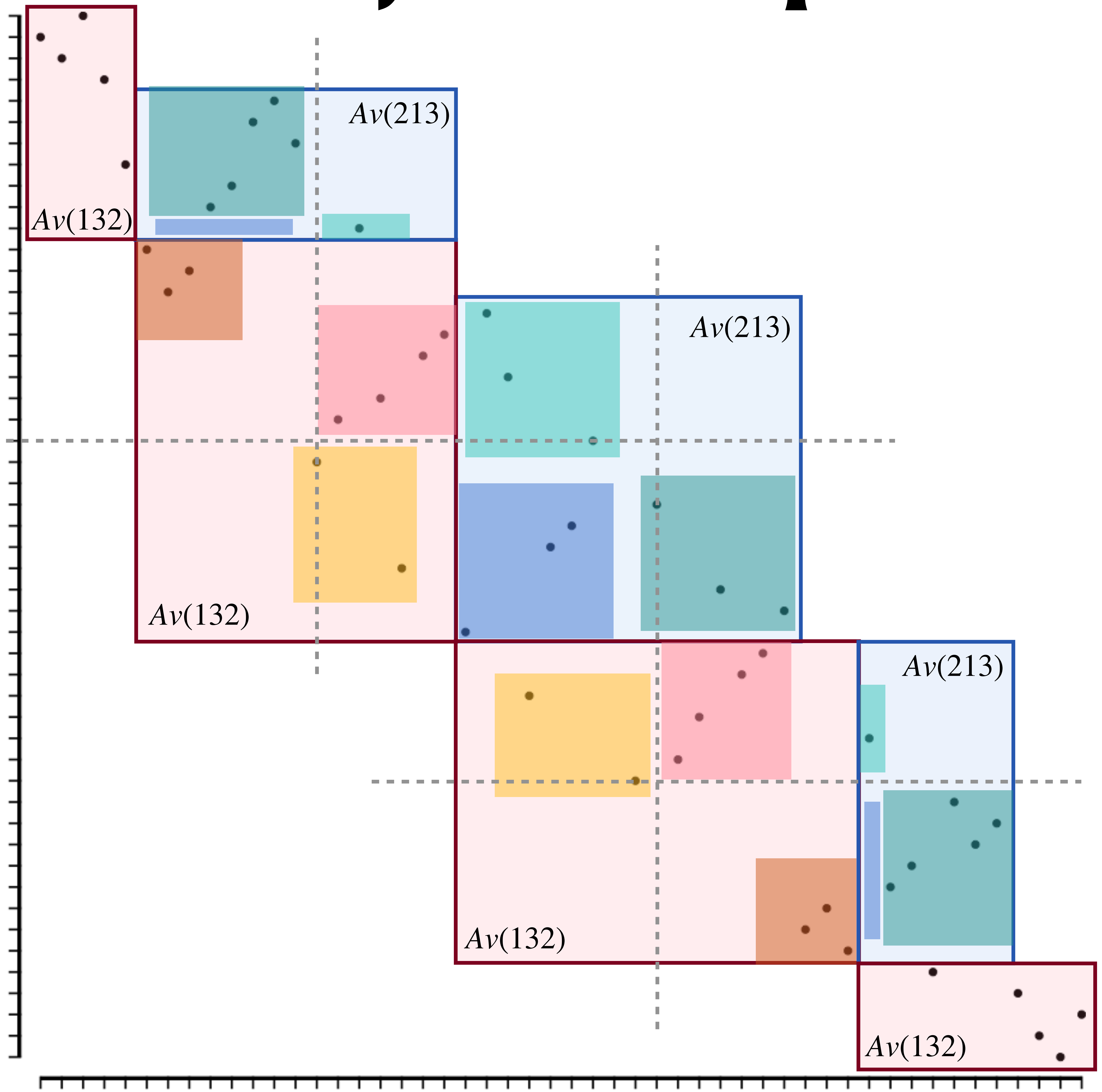

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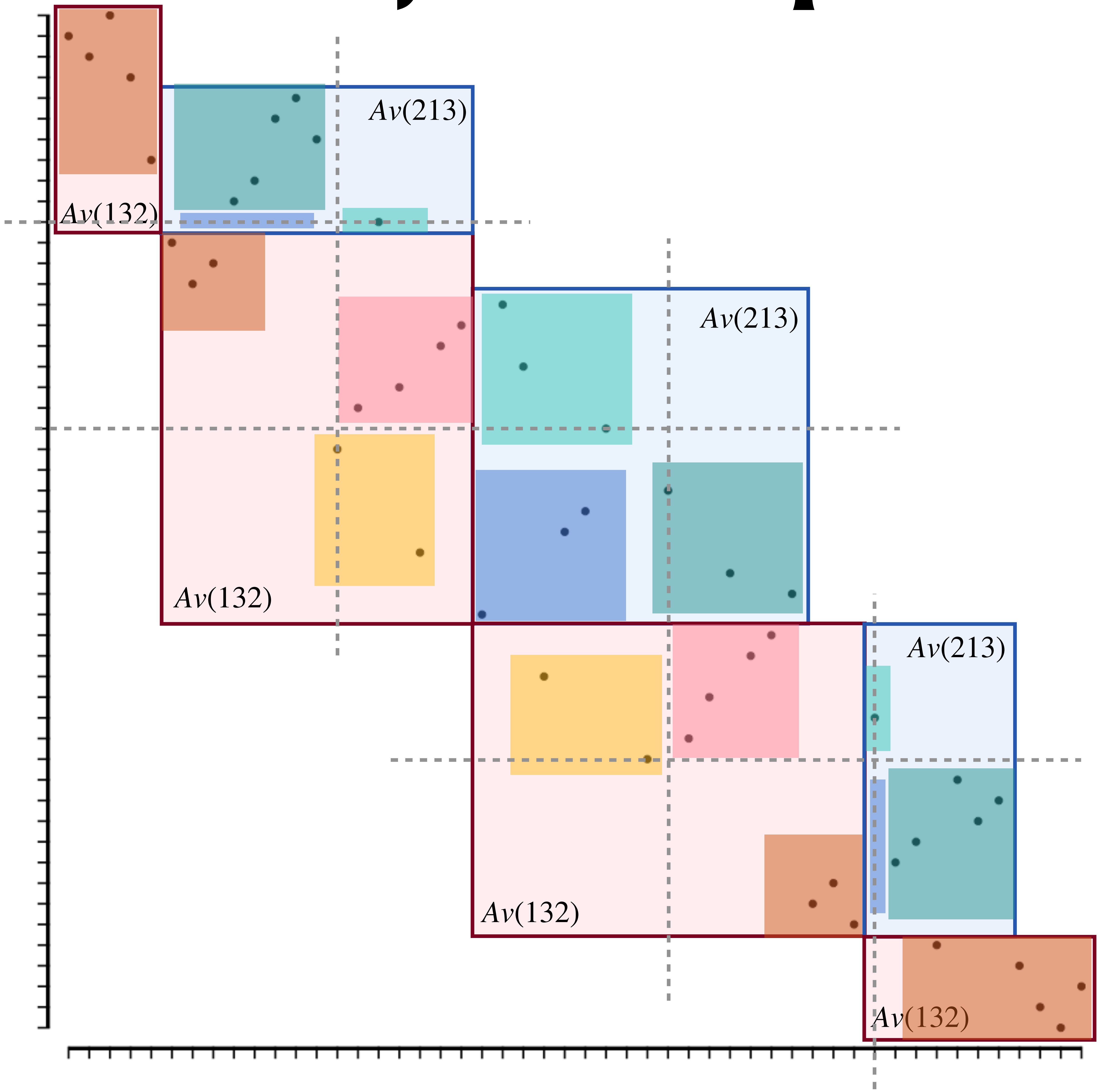

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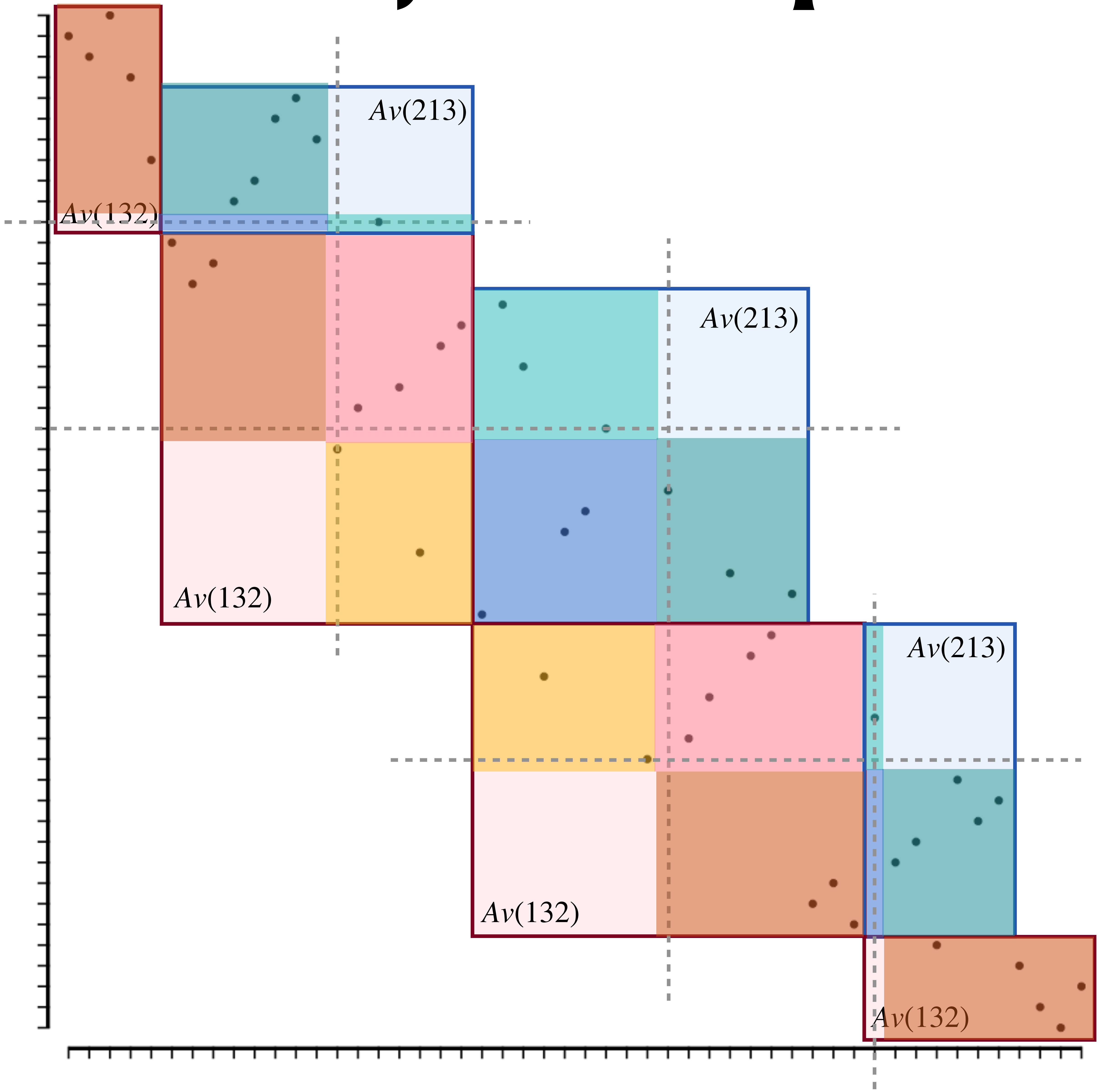

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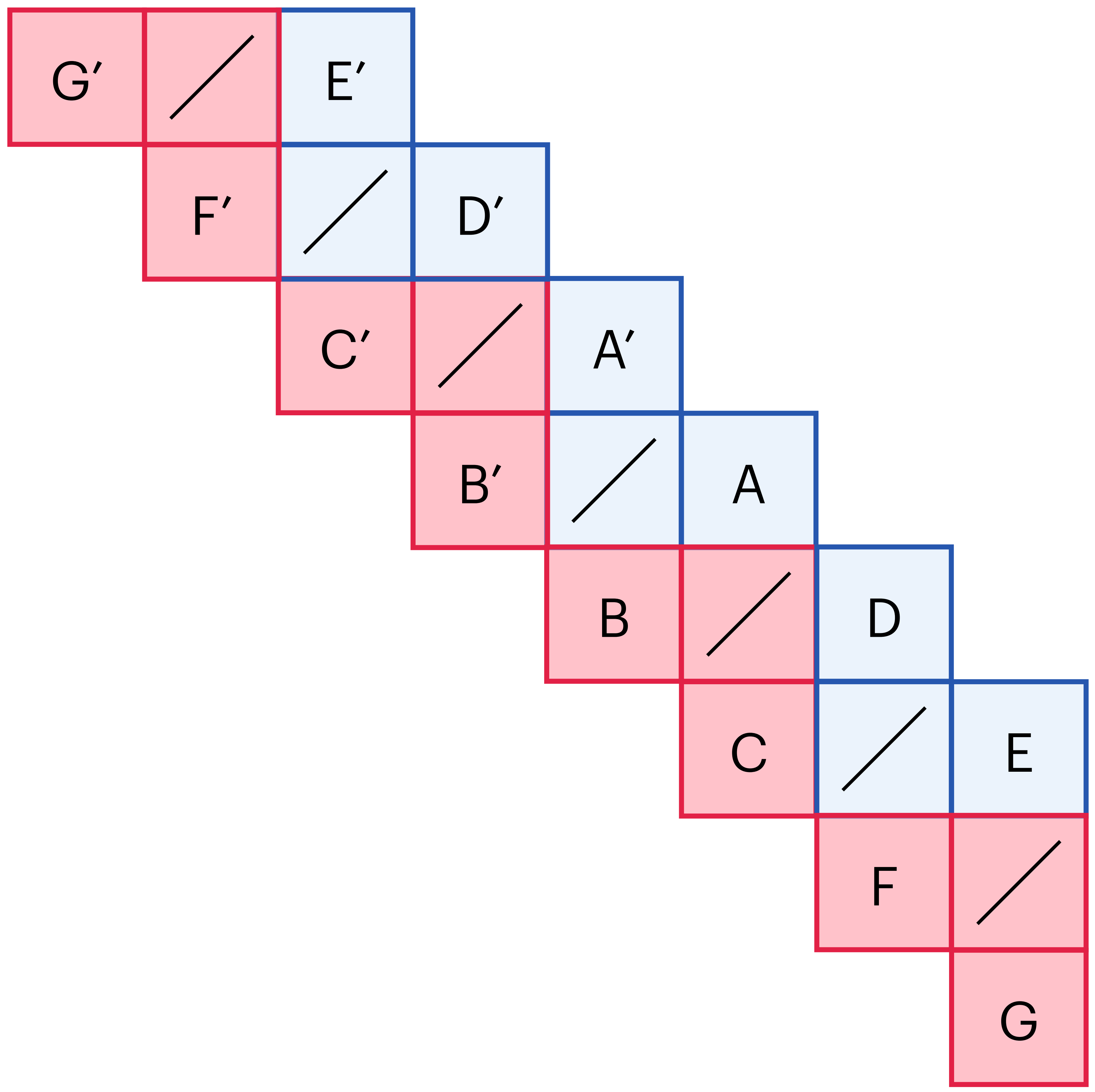
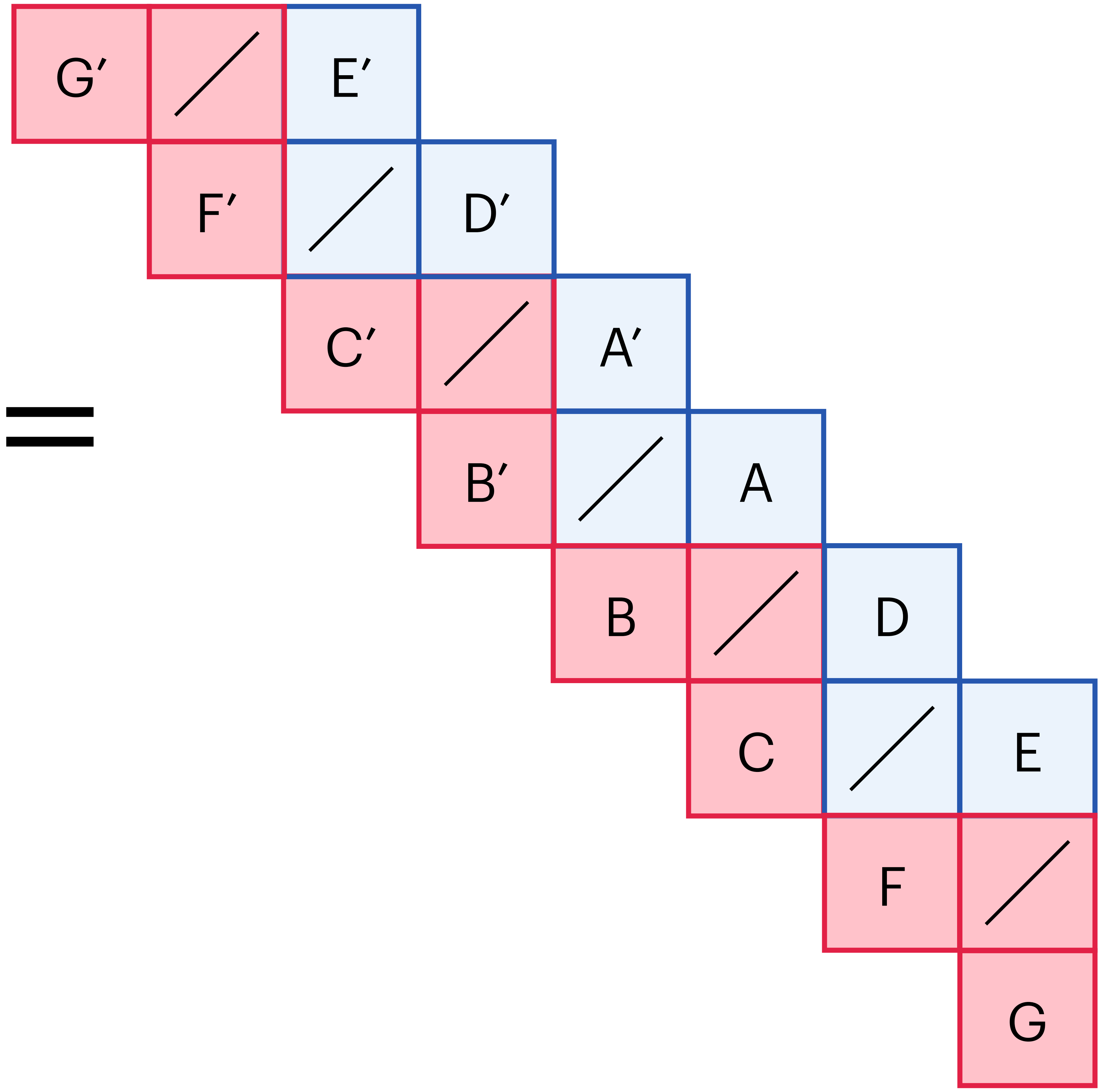
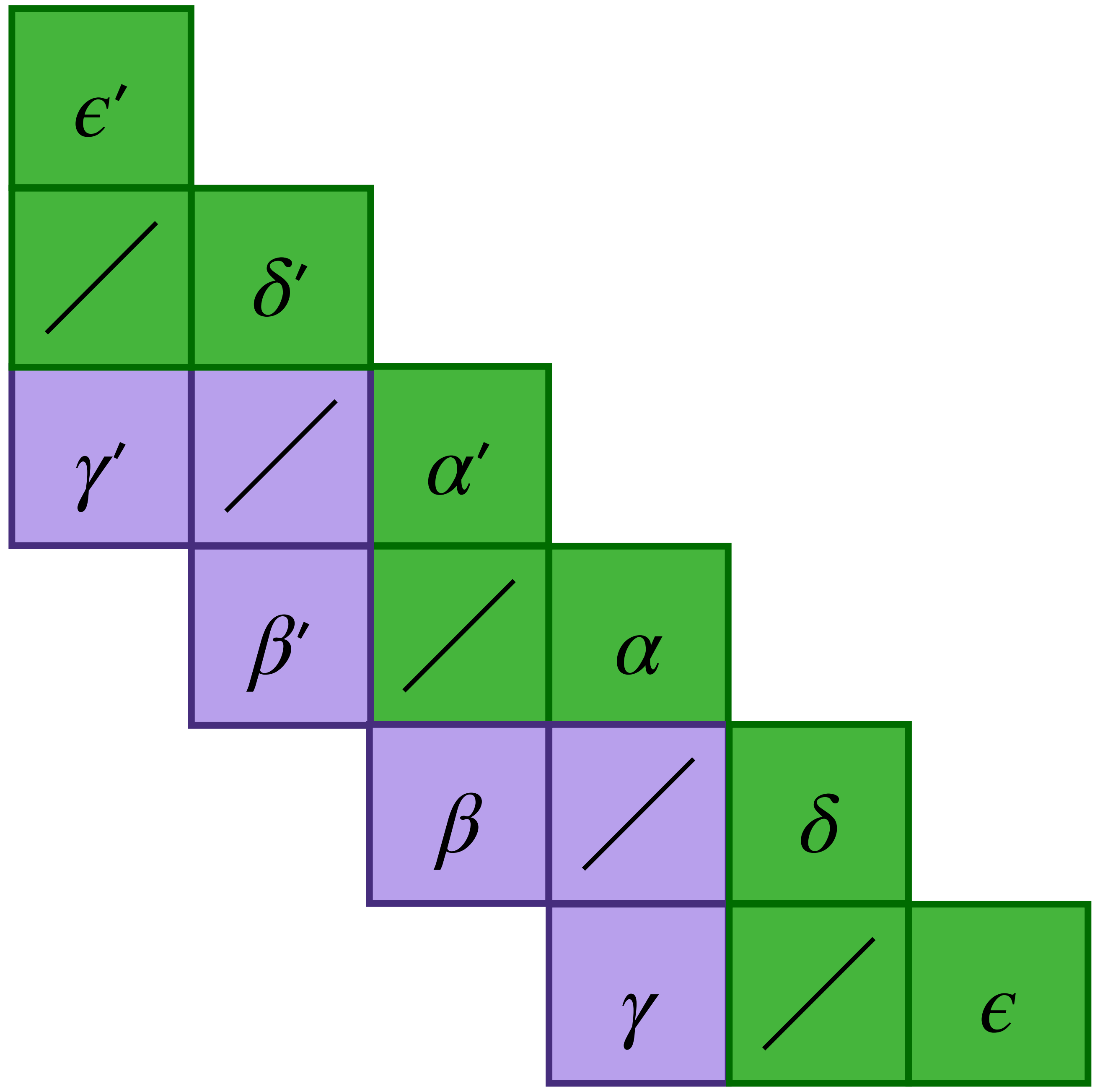


Secondary Decomposition

$\pi =$

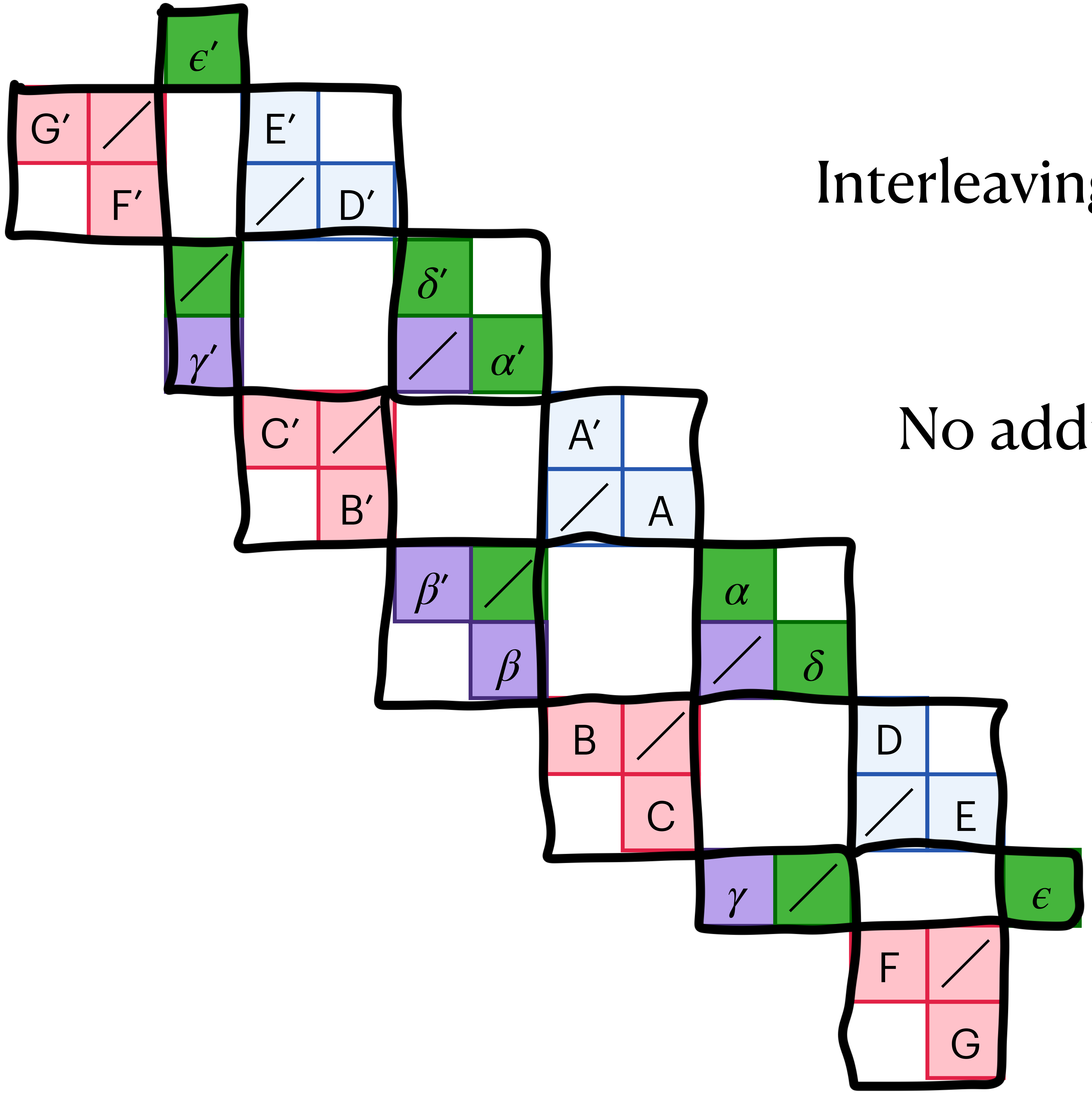


$\sigma =$



Weaving Involutions Together

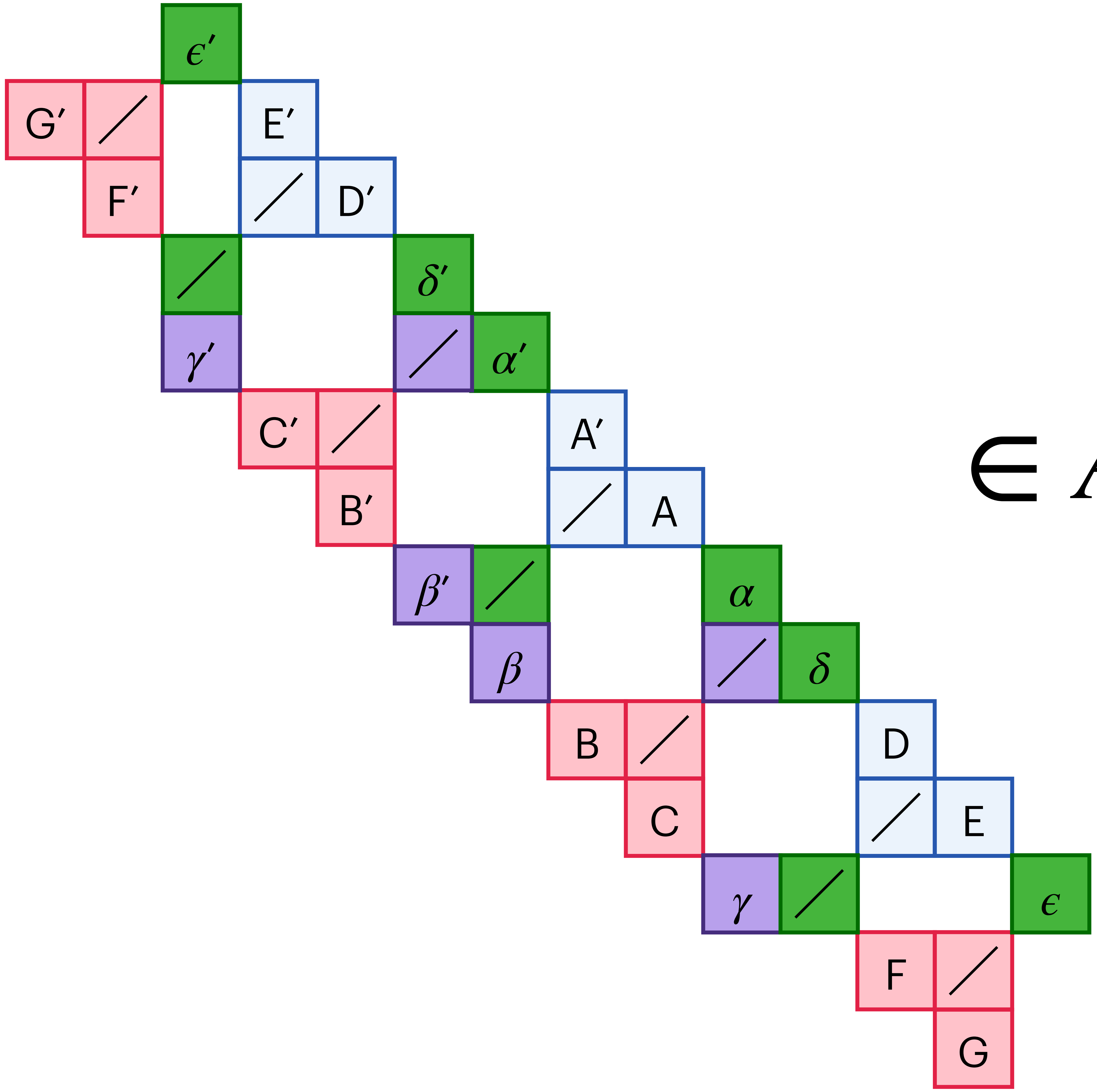
$$\pi \bowtie \sigma =$$



Interleavings are all preserved

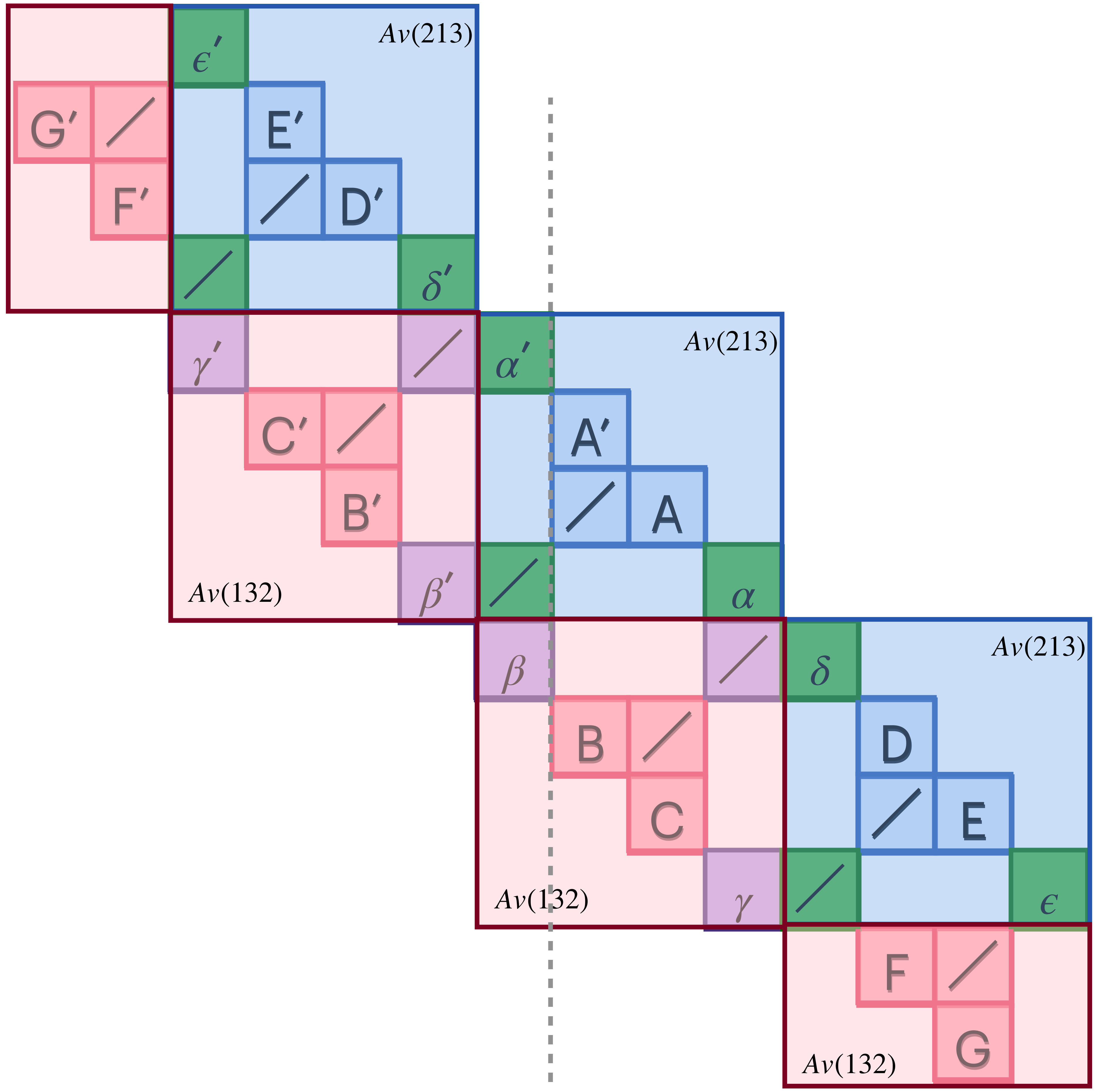
No additional interactions

Weaving Involutions Together

$\pi \bowtie \sigma =$  $\in Av^I(1324)$

Unraveling Involutions

- Let $Av_\circ^I(1324)$ denote the 1324-avoiding involutions with no fixed point that is a right-to-left maximum.
- Then, the map
  $\bowtie : {Av_\circ^I}_m(1324) \times {Av_\circ^I}_n(1324) \longrightarrow {Av_\circ^I}_{m+n}(1324)$
  that we just defined is **injective**.
- This is not obvious!



And so...

- Since
  $\bowtie : Av^I_{\circ\, m}(1324) \times Av^I_{\circ\, n}(1324) \longrightarrow Av^I_{\circ\, m+n}(1324)$
  is injective, it follows that
  $|Av^I_{\circ\, m}(1324)| \cdot |Av^I_{\circ\, n}(1324)| \leq |Av^I_{\circ\, m+n}(1324)|$
- The sequence $a_n = |Av^I_{\circ\, n}(1324)|$ is thus supermultiplicative and so Fekete's Lemma implies that its growth rate $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists.
- What about the sequence we actually care about, $b_n = |Av^I_n(1324)|$? Observe that
  $$a_n \leq b_n \leq a_n + n\, a_{n-1}$$
  and so the growth rate of b_n also exists!