Supertrees

Colin Defant

Princeton University

Joint with Noah Kravitz and Ashwin Sah

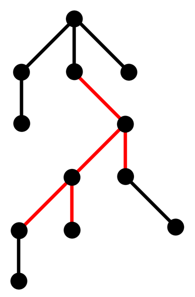
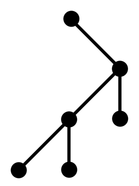
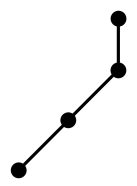
Abstract

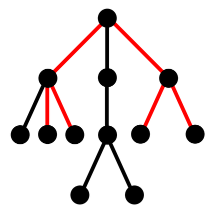
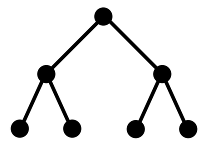
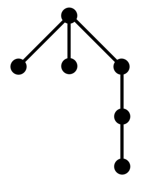
A *k-universal permutation*, or *k-superpermutation*, is a permutation that contains all permutations of length k as patterns. The problem of finding the minimum length of a k-superpermutation has recently received significant attention. One can ask analogous questions for other classes of objects. Here, we study *k-supertrees*. For each $d \geq 2$, we focus on two types of rooted plane trees called d-ary plane trees and $[d]$-trees. Motivated by recent developments in the literature, we consider "contiguous" and "noncontiguous" notions of pattern containment for each type of tree. We obtain both upper and lower bounds on the minimum possible size of a k-supertree in three cases; in the fourth, we determine the minimum size exactly. One of our lower bounds makes use of a recent result of Albert, Engen, Pantone, and Vatter on k-universal layered permutations.

Definitions

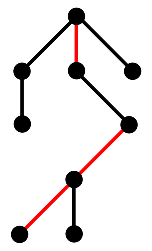
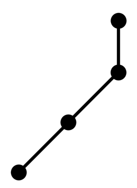
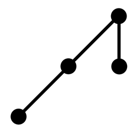
A *d-ary plane tree* is either an empty tree or a root vertex with d subtrees that are linearly ordered from left to right and are themselves d-ary plane trees. A *$[d]$-tree* is a rooted tree in which each vertex has at most d children, which are ordered from left to right.

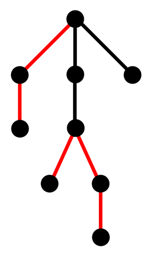
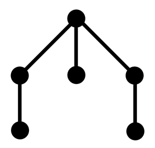
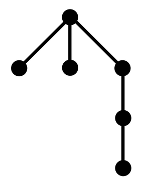
We say a d-ary plane tree (respectively, $[d]$-tree) T^* *contiguously contains* another d-ary plane tree (respectively, $[d]$-tree) T if T is isomorphic to an induced subgraph of T^*. For example,

the 3-ary plane tree  contiguously contains  but not  .

Similarly, the $[3]$-tree  contiguously contains  but not  .

Informally speaking, we say an edge contraction is *legal* if edges do not "overlap" or "cross" during the contraction. We say a d-ary plane tree T^* (respectively, $[d]$-tree) *noncontiguously contains* a d-ary plane tree (respectively, $[d]$-tree) T if we can obtain T from T^* through a sequence of legal edge contractions. For example,

the 3-ary plane tree  noncontiguously contains  but not  .

Similarly, the $[3]$-tree  noncontiguously contains  but not  .

A *contiguous* (respectively, *noncontiguous*) *k-universal d-ary plane tree* is a d-ary plane tree that contiguously (respectively, noncontiguously) contains all d-ary plane trees with k vertices. Let $N^{\text{con}}_{d\text{-ary}}(k)$ (respectively, $N^{\text{non}}_{d\text{-ary}}(k)$) denote the minimum number of vertices in a contiguous (respectively, noncontiguous) k-universal d-ary plane tree. Contiguous and noncontiguous k-universal $[d]$-trees are defined analogously. Let $N^{\text{con}}_{[d]}(k)$ (respectively, $N^{\text{non}}_{[d]}(k)$) denote the minimum number of vertices in a contiguous (respectively, noncontiguous) k-universal $[d]$-tree.

Main Results

Let $\eta_2 = 1$, and let $\eta_d = \frac{1}{2}$ for every $d \geq 3$. In the article [2], we define numbers ρ_d, which arise as reciprocals of roots of certain polynomials; they satisfy $\rho_d = 1 + \frac{4\log d}{d}(1+o(1))$. The main results of the article are the following estimates (where $d \geq 2$ is a fixed integer):

(I) $N^{\text{con}}_{d\text{-ary}}(k) = d^{k-1} + k - 1$;

(II) $\eta_d\, k\log_2(k)(1+o(1)) \leq N^{\text{non}}_{d\text{-ary}}(k) \leq k^{\frac{1}{2}\log_2(k)(1+o(1))}$;

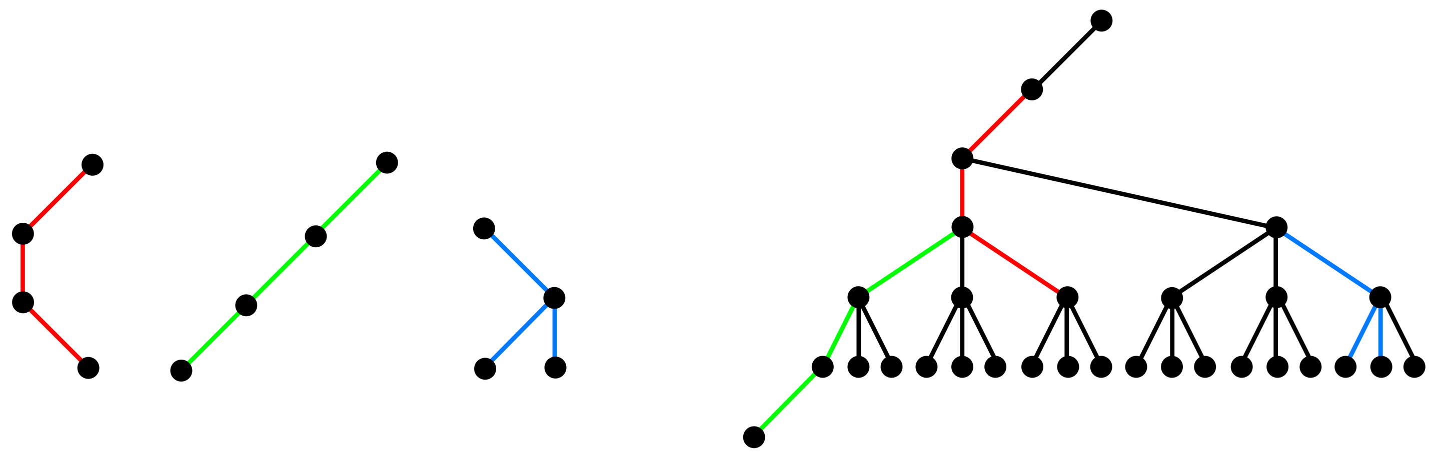
(III) $d^{\frac{k-2}{d}} \leq N^{\text{con}}_{[d]}(k) \leq (\rho_d + o(1))^k$;

(IV) $\frac{\eta_d}{d}k\log_2(k)(1+o(1)) \leq N^{\text{non}}_{[d]}(k) \leq k^{\frac{1}{2}\log_2(k)(1+o(1))}$.

Using (I), one can easily prove that each of these four quantities is at most $d^{k-1}+k-1$. However, the upper bounds in (II), (III), and (IV), which we prove via delicate recursive constructions, greatly improve upon this. In [2], we show that for each fixed d, the quantities $N^{\text{non}}_{d\text{-ary}}(k)$ and $N^{\text{non}}_{[d]}(k)$ differ by at most a constant factor; this explains why (II) and (IV) look similar.

Contiguous d-ary Plane Trees

We prove the exact formula $N^{\text{con}}_{d\text{-ary}}(k) = d^{k-1} + k - 1$ by explicitly constructing minimum-sized contiguous k-universal d-ary plane trees. For $d = 3$ and $k = 4$, this tree is shown on the right. On the left are three 3-ary plane trees with 4 vertices that are contiguously contained in the tree on the right.



Noncontiguous d-ary Lower Bound

The main steps for proving $\eta_d\, k\log_2(k)(1+o(1)) \leq N^{\text{non}}_{d\text{-ary}}(k)$ are:

1. Prove that $N^{\text{non}}_{d\text{-ary}}(k) > \frac{1}{2}N^{\text{non}}_{2\text{-ary}}(k)$ for all $d \geq 3$.
2. Use the postorder traversal to define a bijection ψ from binary (i.e., 2-ary) plane trees to 231-avoiding permutations.
3. Show that if T^* noncontiguously contains T, then $\psi(T^*)$ (classically) contains $\psi(T)$.
4. Deduce that if $\mathbf{T}$ is a k-universal binary plane tree with $N^{\text{non}}_{2\text{-ary}}(k)$ vertices, then $\psi(\mathbf{T})$ contains all 231-avoiding permutations in S_k.
5. Deduce that $\psi(\mathbf{T})$ contains all layered permutations in S_k.
6. Invoke a result of Albert, Engen, Pantone, and Vatter [1], which says that the minimum size of a permutation that contains all layered permutations in S_k is

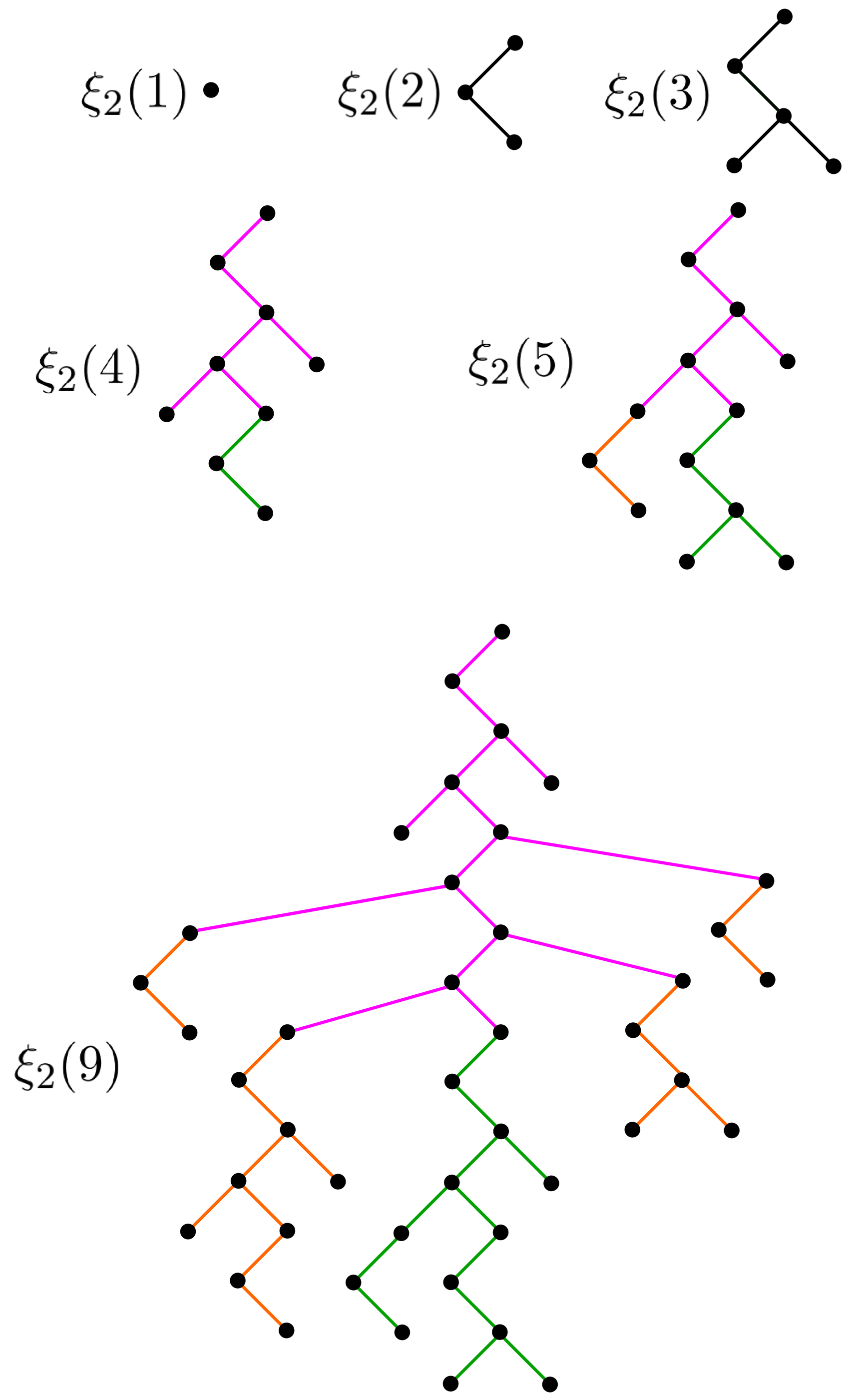
$$(k+1)\lceil\log_2(k+1)\rceil - 2^{\lceil\log_2(k+1)\rceil} + 1 = k\log_2(k)(1+o(1)).$$

Noncontiguous d-ary Upper Bound

Our proof of the subexponential upper bound

$$N^{\text{non}}_{d\text{-ary}}(k) \leq k^{\frac{1}{2}\log_2(k)(1+o(1))}$$

requires a recursive construction of k-universal noncontiguous d-ary plane trees $\xi_d(k)$, some of which are:



Suggestions for Future Work

- Improve the bounds in our main results.
- Does the limit $\lim_{k\to\infty} \frac{N^{\text{non}}_{d\text{-ary}}(k)}{N^{\text{non}}_{[d]}(k)}$ exist, and, if so, what is its value?
- Does the limit $\lim_{k\to\infty} N^{\text{con}}_{[d]}(k)^{1/k}$ exist, and, if so, what is its value?
- Investigate supertrees (i.e., k-universal trees) for noncontiguous containment in nonrooted, nonplane trees. Here, the trees are just viewed as graphs. We say a tree T^* noncontiguously contains a tree T if T can be obtained from T^* by a sequence of edge contractions.
- Investigate supertrees for labeled rooted trees.

References

[1] M. Albert, M. Engen, J. Pantone, and V. Vatter, Universal layered permutations. *Electron. J. Combin.*, **25** (2018).

[2] C. Defant, N. Kravitz, and A. Sah, Supertrees. *Electron. J. Combin.*, **27** (2020).