Counting substructures of highly symmetric structures

Samuel Braunfeld
University of Maryland, College Park
Permutation Patterns 2020

Abstract

We study the asymptotic growth rates of hereditary classes arising as the set of substructures of a highly symmetric (i.e. *homogeneous*) countable structure M. We give a detailed description of the subexponential growth rates, and show how jumps in growth rates correspond to jumps in the structural complexity of M.

The setup

- We will be asymptotically enumerating certain hereditary classes of structures.
- Let M be a countable relational structure, for example a graph, permutation, poset, ordered graph, etc.
- Let $\mathcal{C}$ be the set of (unlabeled) finite substructures of M (up to isomorphism), and let $\mathcal{C}_n$ be the substructures of size n.
- We analyze the asymptotic growth rate of $|\mathcal{C}_n|$, assuming M satisfies the following strong symmetry condition.

Definition. M is *homogeneous* if every finite partial automorphism extends to a full automorphism.

- This is equivalent to the following more algebraic setup: Let G be a group acting on a countable set X. For every $n \in \mathbb{N}$, G acts elementwise on n-subsets of X. We will assume the number of orbits is finite for each n, and enumerate them.
- We are particularly interested in cases where $|\mathcal{C}_n|$ has slow growth, since then M should be highly structured.
- [2] and [4] are nice surveys on and around these topics.

Examples

1. $M = (\mathbb{Q}, \leq)$
   $\mathcal{C}$ consists of linear orders
   $|\mathcal{C}_n| = 1$
2. M is an equivalence relation with infinitely many classes, each infinite
   $\mathcal{C}$ consists of partitions
   $|\mathcal{C}_n| \approx e^{\sqrt{n}}$
3. M is an equivalence relation with infinitely many classes, each of size 2, and a dense linear order on the classes
   $\mathcal{C}$ consists of compositions with each class of size at most 2
   $|\mathcal{C}_n| = Fibonacci(n) \approx \phi^n$ ($\phi \approx 1.618$)
4. $\mathcal{C}$ consists the leaves of full binary trees
   $|\mathcal{C}_n| = Catalan(n) \approx 4^n$
5. $\mathcal{C}$ consists of permutations
   $|\mathcal{C}_n| = n!$
6. $\mathcal{C}$ consists of graphs
   $|\mathcal{C}_n| \approx 2^{n^2}$

Main theorem [1]

We give detailed description of spectrum of growth rates slower than any exponential (in fact slower than ϕ^n), confirming some conjectures of Peter Cameron and Dugald Macpherson.

Theorem. Let M be a countable homogeneous structure, $\mathcal{C}$ its set of unlabeled finite substructures (up to isomorphism), and $\mathcal{C}_n$ its substructures of size n. If $|\mathcal{C}_n| = o\left(\frac{\phi^n}{poly(n)}\right)$ for every polynomial (and $\phi \approx 1.618$), then $|\mathcal{C}_n| = o(c^n)$ for every $c > 1$. Furthermore, one of the following holds.

1. There are $c > 0$, $k \in \mathbb{N}$ such that $|\mathcal{C}_n| \sim cn^k$. ([3])
2. There are $c > 0$, $k \in \mathbb{N}$ such that $|\mathcal{C}_n| = e^{\left(\Theta\left(n^{1-\frac{1}{k}}\right)\right)}$.
3. Let $\log^r(n)$ denote the r-fold iterated logarithm. There are $c > 0$ and $k, r \in \mathbb{N}$ such that $|\mathcal{C}_n| = e^{\left(\Theta\left(\frac{n}{(\log^r(n))^{1/k}}\right)\right)}$.

Corollary. If M is primitive (there is no $Aut(M)$-invariant equivalence relation) and $|\mathcal{C}_n| \not\equiv 1$, then $|\mathcal{C}_n| = \Omega\left(\frac{2^n}{poly(n)}\right)$

Ingredients of the proof

The proof is model-theoretic, but the strategy is very combinatorial. If M encodes certain configurations, its growth rate must be very fast. If it forbids them, we obtain a good structure theory (including a nice tree decomposition), and get subexponential growth rate.

Reduction to the stable case

- M is *stable* if it does not encode an infinite linear order, or equivalently does not encode a copy of the infinite half-graph

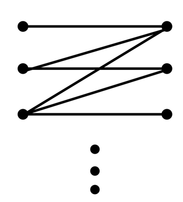

Figure 1: The half-graph

- If M has slower-than-exponential growth rate, there is a stable M^* with the same growth rate [5]

Monadic stability

- M is *monadically stable* if it remains stable under any coloring of its elements. (In particular, any subclass of the finite substructures of M is stable.)
- If M is stable but not monadically stable, it encodes arbitrary bipartite graphs, and so has very fast growth rate.
- If M is monadically stable, it has a nice structure theory. In particular, it has a nice dependence notion, and a tree-decomposition into relatively independent parts.
- Via this structure theory, we prove its growth rate is subexponential.

What do structures with subexponential growth rate look like?

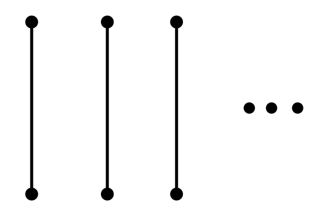

Figure 2: A cellular graph ($K = \emptyset$, each edge is a C_i)

Definition. M is *cellular* if it can be partitioned into finite sets $K \cup \{C_i\}_{i \in \mathbb{N}}$ such that $Aut(M)$ can arbitrarily permute the $C_i s$ while fixing K pointwise.

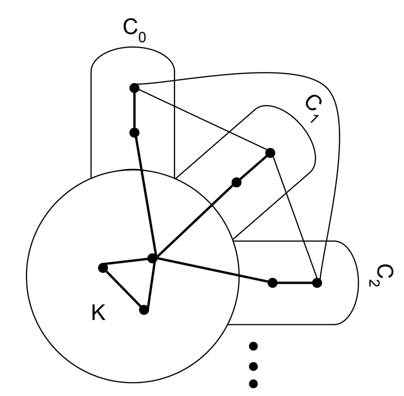

Figure 3: Another cellular graph

- Cellular structures are precisely those with polynomial growth.
- Cellular structures are *hereditarily cellular* of depth 1.
- A hereditarily cellular structure of depth d is defined the same way, but instead of C_i being finite, it is a hereditarily cellular structure of depth $d - 1$.
- For depth 2, the simplest example is that M is an equivalence relation with infinitely many classes, each infinite. (Here, $K = \emptyset$ and each C_i is a single equivalence class.)
- If M has depth 2, it falls under case 2 of the main theorem. If M has depth at least 3, it falls under case 3 (with $r =$ depth -2).
- Thus growth rates are stratified by depth.

A connection to permutation classes

Definition. M is *order-cellular* if it is cellular, except there is a linear order on the $C_i s$. (See 3 in the Examples section.)

- If $\mathcal{C}$ is a permutation class with growth rate less than 2^n, then it essentially arises as the set of substructures of an order-cellular permutation.

Question. One may similarly define *hereditarily order-cellular* structures. Do these similarly account for an interval in the spectrum of growth rates of permutation classes, from 2^n to k^n for some $k \in \mathbb{R}$?

Further work

The exponential range

- The next natural range seems to be when $|\mathcal{C}_n| = o(c^n)$ for some $c \in \mathbb{R}$.
- This will include (generalizations of) geometric grid classes of permutations.
- Here, we expect all the structures to be "tree-like".
- We say M is *NIP* if it does not encode arbitrary bipartite graphs, and *monadically NIP* if it remains NIP after an arbitrary coloring of its elements.

Conjecture. Let M be a countable homogeneous structure, $\mathcal{C}$ its set of unlabeled finite substructures (up to isomorphism), and $\mathcal{C}_n$ its substructures of size n. The following are equivalent.

1. $|\mathcal{C}_n| = o(c^n)$ for some $c \in \mathbb{R}$.
2. M is monadically NIP.
3. $\mathcal{C}$ has no infinite antichains under embeddability. (In fact, this remains true even if the structures in $\mathcal{C}$ are colored arbitrarily.)
4. $\mathcal{C}$ has bounded clique-width.
5. $\mathcal{C}$ is strongly algebraic, i.e. for any hereditary $\mathcal{A} \subseteq \mathcal{C}$, $|\mathcal{A}_n|$ has an algebraic generating function.

- These properties do not align so nicely for general hereditary classes, but the homogeneous setting is much more constrained.

Beyond homogeneity

Question. What can we prove if the homogeneity condition is removed? What about arbitrary hereditary classes?

Conjecture. Let $\mathcal{C}$ be a hereditary class of relational structures. Then either $|\mathcal{C}_n| \sim cn^k$ or grows at least as fast as the partition function.

- This conjecture is known for the case of hereditary graph classes.

References

[1] Samuel Braunfeld. Monadic stability growth rates of ω-categorical structures. *arXiv preprint arXiv:1910.04380*, 2019.

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