The feasible region of consecutive occurrences of permutations is a cycle polytope

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Part 1: Feasible regions and overlap graphs

Goal of the project

Given a permutation τ and a pattern π we set

$$\mathrm{occ}(\pi,\tau) := \#\{\text{ occurrences of } \pi \text{ in } \tau\},$$

$$\mathrm{coc}(\pi,\tau) := \#\{\text{ consecutive occurrences of } \pi \text{ in } \tau\}.$$

The **proportion of (consecutive) occurrences** is defined as

$$\widetilde{\mathrm{occ}}(\pi,\tau) = \frac{\mathrm{occ}(\pi,\tau)}{|\tau|^{|\pi|}}, \quad \widetilde{\mathrm{coc}}(\pi,\tau) = \frac{\mathrm{coc}(\pi,\tau)}{|\tau|}.$$

For a fixed k, a given sequence of permutations $\{\sigma^{(n)}\}_{n\geq 0}$ such that $|\sigma^{(n)}| \to \infty$ gives rise to a **feasible point** $\vec{x} = (x_\pi)_{\pi\in\mathcal{S}_k}$ in $\mathbb{R}^{\mathcal{S}_k}$ defined by

$$x_\pi := \lim_{n\to\infty} \widetilde{\mathrm{coc}}(\pi,\sigma^{(n)}),$$

provided that the limits above exist.

The feasible region of consecutive occurrences

The **feasible region** P_k is the set of feasible points in $\mathbb{R}^{\mathcal{S}_k}$, that is:

$$\left\{\vec{x}\in\mathbb{R}^{\mathcal{S}_k} \middle| \exists\, \sigma^{(n)} \text{ s.t. } \vec{x} = \left(\lim_{n\to\infty}\widetilde{\mathrm{coc}}(\pi,\sigma^{(n)})\right)_{\pi\in\mathcal{S}_k}\right\}$$

We want to answer the following question:

Main question of the project

How does P_k look like?

A simple example

For $k=2$, the sequence $\sigma^{(n)} = 21^{\oplus n}$ gives rise to the limit point $\left(\frac{1}{2},\frac{1}{2}\right)$.

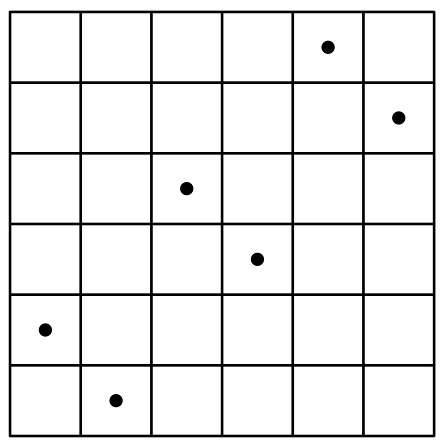

Figure 1: The permutation $\sigma^{(3)}$ from the example above, with $\mathrm{coc}(12,\sigma^{(3)}) = 2$ and $\mathrm{coc}(21,\sigma^{(3)}) = 3$.

A connection with local limits

We recall (see [3]) that a sequence of permutations $\{\sigma^{(n)}\}_{n\geq 0}$ **locally converges** if and only if the sequences

$$\widetilde{\mathrm{coc}}(\pi,\sigma^{(n)}))_{n\geq 0}$$

converge for all $\pi\in\mathcal{S}$.
Moreover, the limiting numbers of the sequences $\widetilde{\mathrm{coc}}(\pi,\sigma^{(n)}))_{n\geq 0}$ *"characterize"* the limiting object of the sequence $\{\sigma^{(n)}\}_{n\geq 0}$.

A characterization of feasible local limits

A description of the feasible regions P_k, for all k, gives a description of all the possible local limits of permutations.

Motivations: the feasible region for classical patterns

The feasible region clP_k for **classical patterns** was first studied in [5] for some particular families of patterns. Describing the region clP_k in full generality is a hard (and probably out of reach) problem.
While the region P_k is related to local limits, the region clP_k corresponds to *permuton limits*. Moreover, the latter somehow generalises the problem of *maximizing (classical) pattern densities*.

The overlap graph

The overlap graph is the central tool that will allow us to decode the structure of P_k.

The overlap graph

The **overlap graph** $\mathcal{O}v(k)$ is an oriented graph on the vertex set $\mathcal{S}_{k-1}$ and such that for each $\pi\in\mathcal{S}_k$ there is an edge from $\mathrm{pat}_{\{1,\dots,k-1\}}(\pi)$ to $\mathrm{pat}_{\{2,\dots,k\}}(\pi)$.

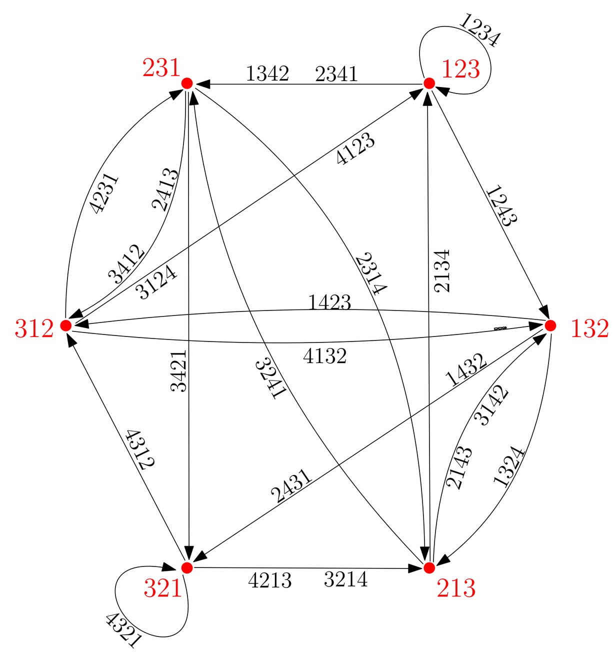

Figure 2: The overlap graph $\mathcal{O}v(4)$.

Part 2: Cycle polytopes and face structure

Cycle polytopes

A simple cycle in an oriented graph is a cycle with no (non-trivial) repeated vertices.

The cycle polytope of a graph

Let G be an oriented graph and $\mathcal{C}$ one of its simple cycles. We define the vector $\vec{a}_{\mathcal{C}}\in\mathbb{R}^{E(G)}$ as

$$(\vec{a}_{\mathcal{C}})_e := |\mathcal{C}|^{-1} \text{ if } e\in\mathcal{C}, \text{ and } (\vec{a}_{\mathcal{C}})_e := 0 \text{ otherwise.}$$

The **cycle polytope** of G is defined as

$$P(G) := \mathrm{conv}\{\vec{a}_{\mathcal{C}} |\ \mathcal{C} \text{ is a simple cycle of } G\}.$$

We have combinatorial characterisations of the geometric properties of $P(G)$.

Theorem (see [1])

If G is strongly connected, then $P(G)$ is a polytope of dimension $\#E(G) - \#V(G)$, given by the equations

$$\sum_{e\in E(G)} x_e = 1,$$

$$\sum_{be(e)=v} x_e = \sum_{en(e)=v} x_e \text{ for all } v\in V(G),$$

where we denote by $be(e)$ (resp. $en(e)$) the beginning (resp. ending) vertex of an edge e.
Furthermore, to each face it corresponds a unique subgraph of G.

Characterization of the feasible region

In the following theorem, we find that the feasible region for consecutive patterns, unlike classical patterns, is a convex polytope.

Theorem (see [1])

The feasible region P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$. From this, it results that $\dim(P_k) = k! - (k-1)!$, and the vertices of P_k correspond to simple cycles in $\mathcal{O}v(k)$.

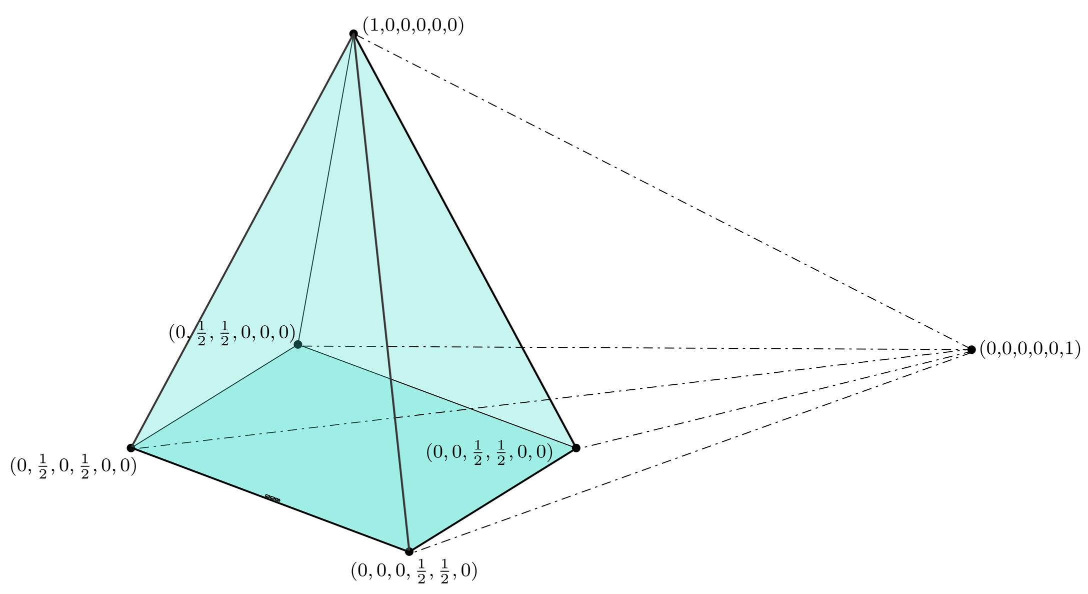

Figure 3: The feasible region for $k=3$ drawn in a 4-dimensional space.

Independence of patterns and consecutive patterns

Theorem (see [1])

The proportion of classical patterns of a certain sequence of permutations imposes no constraints for the proportion of consecutive patterns on the same sequence and vice versa.

This result has been recently strengthen by Bevan (see [4]).

Pattern avoiding permutations

Fix k, and let $\mathcal{A}$ be a *permutation class*. A sequence of permutations $\{\sigma^{(n)}\}_{n\geq 0}\subseteq\mathcal{A}$ such that $|\sigma^{(n)}|\to\infty$ gives rise to an $\mathcal{A}$-feasible point in $\mathbb{R}^{\mathcal{S}_k}$.
We define $P_k^{\mathcal{A}}$ as the set of $\mathcal{A}$-feasible points in $\mathbb{R}^{\mathcal{S}_k}$.

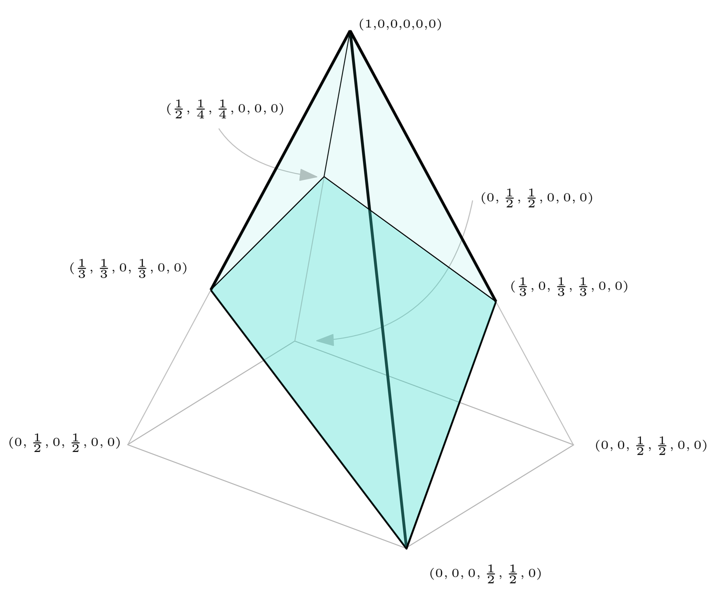

Figure 4: The Av(321)-feasible region for $k=3$.

Partial results for: $\mathrm{Av}(\rho)$ with $|\rho|=3$ and $\mathrm{Av}(n\dots 1)$ for any $n\geq 2$, see [2].

References

[1] Borga J. and Penaguiao R., The feasible region for consecutive patterns of permutations is a cycle polytope, ArXiv preprint: 1910.02233 (2019).
[2] Borga J. and Penaguiao R., The feasible region for consecutive patterns of some permutation classes is a cycle polytope, in preparation.
[3] Borga J., Local convergence for permutations and local limits for uniform ρ-avoiding permutations with $|\rho|=3$, *Probability Theory and Related Fields* 176.1-2 (2020): 449-531.
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[5] Richard Kenyon, Daniel Král, Charles Radin, and Peter Winkler, Permutations with fixed pattern densities, *Random Structures Algorithms*, 56(1):220–250, (2020).