Pattern Avoiding Permutations and Involutions with a Unique Longest Increasing Subsequence

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Preliminaries

Let p be a permutation. p has a unique longest increasing subsequence (ULIS) if p has an increasing subsequence that is longer than all other increasing subsequences.

In this poster, we investigate the number of permutations which avoid a given pattern q of length 3 and have a ULIS, with the more trivial results stated briefly in the "Other Results" section. As the total number of permutations avoiding a pattern q of length 3 are counted by the well-known Catalan numbers, it is surprising that counting such permutations with a ULIS can be quite difficult.

If p has a ULIS, then its inverse, p^{-1}, has a ULIS, and so does its reverse complement. Therefore the six choices of length 3 patterns q can be reduced to 4: 123,231,132, and 321. Let $u_n(q)$ denote the number of permuations avoiding q of length n with a ULIS, and let $u_q(z) = \sum_{n \geq 0} u_n(q) z^n$ be the ordinary generating function, with $u_0(q) = 1$. Similarly, denote by $i_n(q)$ the number of involutions avoiding q of length n with a ULIS.

The Pattern 231

Let $p = p_1 p_2 \cdots p_n$ be a permutations avoiding the pattern 231. If $p_i = n$, we can set $L = p_1 p_2 \cdots p_{i-1}$ and $R = p_{i+1}, p_{i+2}, \cdots, p_n$. Note all entries in L are greater than all entries of R. If p has a ULIS, then L and R must both have a ULIS. If L and R have a ULIS, the only case in which p does not have a ULIS is when R contains exactly one entry. This leads to the following identity:

$$u_{231}(z) = u_{231}(z) z (u_{231}(z) - z) + 1.$$

This gives

$$u_{231}(z) = \frac{1 + z^2 - \sqrt{1 - 4z + 2z^2 + z^4}}{2z}.$$

The dominant singularity is 0.2956, giving us that $\lim_{n\to\infty}(u_n(231))^{1/n} \approx 3.383.$

A pattern is indecomposable if it cannot be cut into two parts so that every entry before the cut is smaller than every entry after the cut. Notice that 231 is indecomposable. Although the following theorem was unnecessary in our investigation above, it will be useful later.

Theorem 1

If q is an indecomposable pattern, then $lim_{n\to\infty}(u_n(q))^{1/n}$ exists.

The Pattern 132

Permutations

There is a well-known bijection ψ from 132-avoiders of size n to plane rooted unlabeled trees on $n+1$ vertices. The number of longest increasing subsequences of a 132-avoider p is then the number of leaves of max distance from the root of $\psi(p)$. The following theorem is proven in [3].

Theorem 2

Let $a_{n,k}$ be the probability a randomly selected plane rooted unlabeled tree on n vertices has k leaves at maximum distance from the root. Then

$$lim_{n\to\infty} a_{n,k} = 2^{-k}.$$

Therefore $\lim_{n\to\infty} \frac{u_n(132)}{C_n} = \frac{1}{2}.$

The first few terms of the sequence $u_n(132)$, given in the OEIS as sequence A152880, lead to the following conjectures:

1. $u_n(132)/C_n \geq 0.5$ for all n.
2. The sequence $u_n(132)/C_n$ monotone decreasing for $n \geq 3$.

Notice the truth of the second statement would imply the first.

The Pattern 132 - Involutions

Theorem 2

The number of 132-avoiding involutions of length n is equal to the number of lattice paths π consisting of $n+1$ steps (each of which are U steps or D steps) so that each prefix and each suffix of π contains strictly more U steps than D steps.

In [2], the lattice paths described above are called bidirectional ballot sequences. Letting B_n the the number of such sequences of length n, [2] tells us that

$$i_n(132) = B_{n+1} \sim \frac{2^{n+1}}{4(n+1)} = \frac{2^{n-1}}{n+1}.$$

Since the total number of involution avoiding 132 is $\binom{n}{\lfloor n/2 \rfloor} \sim \sqrt{\frac{2}{\pi}} \frac{2^n}{\sqrt{n}}$, we have the probability of a (uniformly) randomly selected 132-avoiding involution having a ULIS being about $\frac{c}{\sqrt{n}}$.

The RS Correspondence

The Robinson-Schensted (RS) correspondence is a bijection from all permutations of length n to pairs of Standard Young Tableaux (SYT) of the same shape on n boxes. The RS-bijection restricted to permutations avoiding 321 maps onto pairs of SYT of the same shape that have at most 2 rows.

The Pattern 321

First, it is known that the number of SYT of shape (m,m) (meaning each of the 2 rows has m boxes) is C_m. Thus the number of pairs of SYT of shape (m,m) is C_m^2.

We use a bijection of Claesson and Kitaev [1] to provide a lower bound for $u_{2m+1}(321)$. Call this bijection f, which sends the set of all 321-avoiding permutations of length n to into the set of indecomposable 321-avoiding permutations of length $n+1$. The steps of this bijection are outlined below with an example.

35124786	(1) Begin by underlining all left-to-right maxima to the right of 1 that are not right-to-left minima
359124786	(2) Insert $n+1$ directly before 1 and underline
357124896	(3) Cyclically permute underlined entries to left

Theorem 3

If p is a permutation of shape (m,m), then $f(p)$ has a ULIS.

Since f is injective, the number of permutations of shape $(m+1,m)$ with a ULIS is at least the number of permutations of shape (m,m), which is

$$C_m^2 = \frac{\binom{2m}{m}^2}{(m+1)^2} \sim \frac{4^{2m}}{m^3 \pi} \sim \frac{2}{\pi}\frac{4^n}{n^3}$$

where $n = 2m+1$. This, along with Theorem 1, gives that $\lim_{n\to\infty}(u_n(321))^{1/n} = 4.$

By way of contradiction, it can also be shown that the generating function $u_{321}(x)$ is not rational.

References

[1] Claesson, A.; Kitaev, S., Classification of bijections between 321- and 132-avoiding permutations. *Sém. Lothar. Combin.* **60** (2008/09), Art. B60d, 30 pp.

[2] Hackl, B.; Heuberger, C.; Prodinger, H.; Wagner, S., Analysis of bidirectional ballot sequences and random walks ending in their maximum. *Ann. Comb.* **20** (2016), no. 4, 775–797.

[3] Harris, T. E., A Theory of Branching Processes, *Prentice Hall* Englewood Cliffs, NJ, 1963,