Counting Permutations by Peaks, Descents, and Cycle Type

Ira M. Gessel[1] Yan Zhuang[†2]

[1]Department of Mathematics, Brandeis University

[2]Department of Mathematics and Computer Science, Davidson College

Abstract

We derive a general formula describing the joint distribution of two permutation statistics—the peak number and the descent number—over any set of permutations whose quasisymmetric generating function is a symmetric function. Our formula involves a certain kind of plethystic substitution on quasisymmetric generating functions. We apply this result to cyclic permutations, involutions, and derangements, and to give a generating function formula for counting permutations by peaks, descents, and cycle type. We recover as special cases results previously derived by Désarménien–Foata, Gessel–Reutenauer, Fulman, and Diaconis–Fulman–Holmes.

Background: Permutation Statistics

- Let $\mathfrak{S}_n$ denote the symmetric group of permutations of $[n] := \{1,2,\dots,n\}$.
- For a permutation $\pi \in \mathfrak{S}_n$, we call i a *descent* of π if $\pi(i) > \pi(i+1)$ and a *peak* of π if $\pi(i-1) < \pi(i) > \pi(i+1)$.
- Let $\operatorname{des}(\pi)$ denote the number of descents of π, $\operatorname{maj}(\pi)$ the sum of all descents of π, and $\operatorname{pk}(\pi)$ the number of peaks of π.
- Every permutation can be uniquely decomposed into a sequence of maximal increasing consecutive subsequences, called *increasing runs*.
- The *descent composition* of π, denoted $\operatorname{Comp}(\pi)$, is the composition whose parts are the increasing run lengths of π.
- Let $A_n(t) := \sum_{\pi\in\mathfrak{S}_n} t^{\operatorname{des}(\pi)+1}$ be the nth *Eulerian polynomial*.

Background: Symmetric Functions

- Let $x_1, x_2, \dots$ be commuting variables.
- For a positive integer n, let $p_n := \sum_{i=1}^\infty x_i^n$.
- For a partition $\lambda = (\lambda_1,\lambda_2,\dots\lambda_k)$, let $p_\lambda := p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_k}$.
- The $\mathbb{Q}$-linear span of $\{p_\lambda\}$ is the $\mathbb{Q}$-algebra Λ of *symmetric functions*.
- For a composition $L = (L_1, L_2, \dots, L_k)$, let $\operatorname{Des}(L) := \{L_1, L_1+L_2, \dots, L_1+\cdots+L_{k-1}\}$ and let
$$F_L := \sum_{\substack{i_1\le i_2\le\cdots\le i_n \\ i_j < i_{j+1} \text{ if } j\in\operatorname{Des}(L)}} x_{i_1}x_{i_2}\cdots x_{i_n}.$$
- For a set $\Pi \subseteq \mathfrak{S}_n$, let $Q(\Pi) := \sum_{\pi\in\Pi} F_{\operatorname{Comp}(\pi)}$; this is the *quasisymmetric generating function* of Π.

Gessel–Reutenauer Formula

- **Theorem** (Gessel–Reutenauer [3])**:** Let $\Pi \subseteq \mathfrak{S}_n$. If $Q(\Pi)\in\Lambda$, then
$$\frac{\sum_{\pi\in\Pi} t^{\operatorname{des}(\pi)+1} q^{\operatorname{maj}(\pi)}}{(1-t)(1-tq)\cdots(1-tq^n)} = \sum_{k=0}^\infty \operatorname{ps}_k(Q(\Pi))\,t^k \tag{1}$$
where $\operatorname{ps}_k(f) := f(1,q,\dots,q^{k-1})$.
- Generalizes classical formula of MacMahon for joint distribution of des and maj over $\mathfrak{S}_n$.
- Gessel–Reutenauer [3] used (1) to derive formulas for joint distribution of des and maj over cyclic permutations, involutions, and derangements.
- Fulman [2] used (1) to derive a formula for joint distribution of des, maj, and cycle type over $\mathfrak{S}_n$.
- **Our main result:** Analogue of (1) for joint distribution of pk and des.

Plethysm

- Let A be a $\mathbb{Q}$-algebra of formal power series in some set of variables.
- The *plethysm* operation $\Lambda\times A \to A$, $(f,a)\mapsto f[a]$, is defined by:
  - For any $i\ge 1$, $p_i[a]$ is the result of replacing each variable in a with its ith power;
  - For any $a\in A$, the map $f\mapsto f[a]$ is a $\mathbb{Q}$-algebra homomorphism.

Main Result

- For $\Pi\subseteq\mathfrak{S}_n$, define $P^{(\mathrm{pk},\mathrm{des})}(\Pi;y,t) := \sum_{\pi\in\Pi} y^{\operatorname{pk}(\pi)+1}t^{\operatorname{des}(\pi)+1}$.
- For $f\in\Lambda$, a variable y, and $k\in\mathbb{Z}$, define $\Theta_{y,k}(f) := f[k(1-\alpha)]|_{\alpha=-y}$.
- **Theorem:** Let $\Pi\subseteq\mathfrak{S}_n$. If $Q(\Pi)\in\Lambda$ and $Q(\Pi)=\sum_{\lambda\vdash n} c_\lambda p_\lambda$, then
$$\frac{1}{1+y}\left(\frac{1+yt}{1-t}\right)^{n+1} P^{(\mathrm{pk},\mathrm{des})}\left(\Pi;\frac{(1+y)^2t}{(y+t)(1+yt)},\frac{y+t}{1+yt}\right) = \sum_{k=0}^\infty \Theta_{y,k}(Q(\Pi))\,t^k = \sum_{\lambda\vdash n} c_\lambda \frac{A_{l(\lambda)}(t)}{(1-t)^{l(\lambda)+1}}\prod_{k=1}^{l(\lambda)}(1-(-y)^{\lambda_k}). \tag{2}$$
(Here $l(\lambda)$ is the number of parts of λ and λ_k is the kth part of λ.)
- Generalizes formula of Z. [6] for joint distribution of pk and des over $\mathfrak{S}_n$.

Application: Cyclic Permutations

- Let $\mathfrak{C}_n$ be the set of permutations with cycle type (n).
- Let μ be the number-theoretic Möbius function.
- **Theorem:** We have
$$P^{(\mathrm{pk},\mathrm{des})}\left(\mathfrak{C}_n;\frac{(1+y)^2t}{(y+t)(1+yt)},\frac{y+t}{1+yt}\right) = \frac{(1+y)}{n(1+yt)^{n+1}}\sum_{d|n}\mu(d)(1-(-y)^d)^{n/d}(1-t)^{n-n/d}A_{n/d}(t). \tag{3}$$
- Setting $y=0$ in (3) recovers formula of Gessel–Reutenauer [3] for distribution of des over $\mathfrak{C}_n$.

Application: Peaks, Descents, and Cycle Type

- Define $F_n^{(\mathrm{pk},\mathrm{des})}(y,t,z_1,z_2,\dots) := \sum_{\pi\in\mathfrak{S}_n} y^{\operatorname{pk}(\pi)+1}t^{\operatorname{des}(\pi)+1}\prod_{i=1}^\infty z_i^{N_i(\pi)}$, where $N_i(\pi)$ is the number of i-cycles in π.
- **Theorem:** We have
$$\frac{1}{1-t}+\frac{1}{1+y}\sum_{n=1}^\infty\left(\frac{1+yt}{1-t}\right)^{n+1} F_n^{(\mathrm{pk},\mathrm{des})}\left(\frac{(1+y)^2t}{(y+t)(1+yt)},\frac{y+t}{1+yt},z_1,z_2,\dots\right)x^n = \sum_{k=0}^\infty t^k\prod_{i=1}^\infty \exp\left(\sum_{m_i=1}^\infty \frac{(z_ix^i)^{m_i}}{im_i}\sum_{d|i}\mu(d)(k(1-(-y)^{dm_i}))^{i/d}\right). \tag{4}$$
- Setting $y=0$ in (4) recovers formula of Fulman [2] for joint distribution of des and cycle type over $\mathfrak{S}_n$.
- Setting $y=1$ in (4) recovers formula of Diaconis–Fulman–Holmes [1] for joint distribution of pk and cycle type over $\mathfrak{S}_n$.

Conclusion

- See our paper [4] for other applications of (2), as well as plethystic formulas for other permutation statistics and their applications.
- **Why do we care?** Statistics like des and pk encode properties of a permutation in one-line representation; generally very difficult to study distributions of such statistics while refining by cycle structure.
- **What's next?** Various pattern avoidance classes have symmetric quasisymmetric generating functions [5]. Can use (1), (2), and other plethystic formulas to study distributions of statistics over these classes.

References

[1] Persi Diaconis, Jason Fulman, and Susan Holmes, *Analysis of casino shelf shuffling machines*, Ann. Appl. Probab. **23** (2013), no. 4, 1692–1720.

[2] Jason Fulman, *The distribution of descents in fixed conjugacy classes of the symmetric groups*, J. Combin. Theory Ser. A **84** (1998), no. 2, 171–180.

[3] Ira M. Gessel and Christophe Reutenauer, *Counting permutations with given cycle structure and descent set*, J. Combin. Theory Ser. A **64** (1993), no. 2, 189–215.

[4] Ira M. Gessel and Yan Zhuang, *Plethystic formulas for permutation enumeration*, Preprint, https://arxiv.org/abs/2001.00654, 2020.

[5] Zachary Hamaker, Brendan Pawlowski, and Bruce E. Sagan, *Pattern avoidance and quasisymmetric functions*, Algebr. Comb. **3** (2020), no. 2, 365–388.

[6] Yan Zhuang, *Eulerian polynomials and descent statistics*, Adv. in Appl. Math. **90** (2017), 86–144.