Bijective Proofs of Shuffle Compatibility

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Permutation Patterns
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If π, σ are permutations with $\pi \cap \sigma = \emptyset$ then their *shuffle set* is

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$$25 \sqcup\!\sqcup\, \mathbf{74} = \{25\mathbf{74},\ 2\mathbf{7}5\mathbf{4},\ 2\mathbf{74}5,\ \mathbf{7}25\mathbf{4},\ \mathbf{7}2\mathbf{4}5,\ \mathbf{74}25\}.$$

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$$\begin{aligned}\operatorname{Des}(25 \sqcup\!\sqcup\, \mathbf{74}) &= \{\{\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\}\ \}\} \\ &= \operatorname{Des}(12 \sqcup\!\sqcup\, \mathbf{43}).\end{aligned}$$

History.

who	when	what
Stanley	1972	implicit in P-partitions
Gessel and Zhuang	2018	explicit using shuffle algebras
Grinberg	2018	enriched P-partitions
Oğuz	2018	conjecture of Gessel and Zhuang
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Given π, π' with $|\pi| = |\pi'|$ and σ there is a *fundamental bijection*

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Ex. If $\pi = \mathbf{1423}$, $\pi' = \mathbf{2314}$ and $\sigma = 756$ then

$$\Phi(\mathbf{17542}6\mathbf{3}) = \mathbf{27531}6\mathbf{4}.$$

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Corollary

Des *is shuffle compatible.*

THANKS FOR

LISTENING!