Pattern-avoiding permutations by inactive sites

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Active & inactive sites

Active sites of a permutation in $\mathcal{S}_n(\sigma)$ are positions in which $n+1$ can be inserted to get a permutation in $\mathcal{S}_{n+1}(\sigma)$. An *inactive site* is a site that is not active.

Example. $\pi = 45132 \in \mathcal{S}_5(123)$ has 6 sites:

$${}_\circ 4 {}_\circ 5 {}_\circ 1 {}_\circ 3 {}_\circ 2 {}_\circ$$

$\mathbf{6}45132$, $4\mathbf{6}5132 \in \mathcal{S}_6(123)$, but inserting 6 in any other site creates a 123 pattern. Thus π has 2 active and 4 inactive sites:

$${}_1 4 {}_2 5 {}_\times 1 {}_\times 3 {}_\times 2 {}_\times.$$

Let $\mathcal{B}_{n,m}(\sigma)$ be the set of permutations in $\mathcal{S}_n(\sigma)$ with exactly m **inactive** sites, and let $b_{n,m} = |\mathcal{B}_{n,m}(\sigma)|$.

Patterns of size 3

For $\sigma \in \mathcal{S}_3$, the set $\mathcal{S}_n(\sigma)$ contains only one element with $n+1$ active sites. Every other permutation in $\mathcal{S}_n(\sigma)$ has at least two active sites, so $b_{n,0} = 1$ and $\mathcal{B}_{n,n}(\sigma) = \mathcal{B}_{n,n+1}(\sigma) = \varnothing$.

Theorem. For $n > 1$, $m < n$, and every $\sigma \in \mathcal{S}_3$,

$$b_{n,m} = b_{n,m-1} + b_{n-1,m},$$

where $b_{n,0} = 1$ and $b_{n,n} = b_{n,n+1} = 0$. Thus

$$b_{n,m} = \frac{n-m}{n}\binom{n+m-1}{n-1}.$$

This gives the Catalan triangle A009766 in the OEIS.

The recurrence relation for $b_{n,m}$ can be shown by writing $\mathcal{B}_{n,m}(\sigma)$ as a disjoint union of sets split according to the position of n in the permutation.

For example, for $\sigma = 132$, let $\mathcal{B}^{n*}$ be the set of permutations in $\mathcal{B}_{n,m}(132)$ of the form $n \ominus \pi'$ with $\pi' \in \mathcal{S}_{n-1}(132)$.

$$n \ominus \pi' \in \mathcal{B}_{n,m}(132) \text{ if and only if } \pi' \in \mathcal{B}_{n-1,m}(132).$$

Thus $|\mathcal{B}^{n*}| = b_{n-1,m}$. The first and last sites of π' are active relative to 132. If $I(\pi')$ is the number of inactive sites, then

$$I(\mathrm{ch}_{i+1}(\pi')) = I(\mathrm{ch}_i(\pi')) + 1,$$

where $\mathrm{ch}_i(\pi')$ is the permutation in $\mathcal{S}_n(\sigma)$ obtained by inserting n into the i-th active site of π', labeled from left to right.

For $\pi \in \mathcal{B}_{n,m-1}(132)$, let π' and i be such that $\pi = \mathrm{ch}_i(\pi')$. Define $\gamma(\pi) = \mathrm{ch}_{i+1}(\pi')$. E.g. $\pi' = 645312$ has 5 active sites

$${}_0 6 {}_1 4 {}_\times 5 {}_2 3 {}_3 1 {}_\times 2 {}_4,$$

and we have

$$\mathbf{7}645312 \overset{\gamma}{\mapsto} 6\mathbf{7}45312 \overset{\gamma}{\mapsto} 645\mathbf{7}312 \overset{\gamma}{\mapsto} 6453\mathbf{7}12 \overset{\gamma}{\mapsto} 645312\mathbf{7}.$$

The map $\gamma : \mathcal{B}_{n,m-1}(132) \to \mathcal{B}_{n,m}(132) \setminus \mathcal{B}^{n*}$ is bijective, and so $|\mathcal{B}_{n,m}(132) \setminus \mathcal{B}^{n*}| = b_{n,m-1}$.

Patterns of size 4

At most one inactive site

Theorem. Let $C_n = \frac{1}{n+1}\binom{2n}{n}$. For every $\sigma \in \mathcal{S}_4$, we have

$$|\mathcal{B}_{n,0}(\sigma)| = C_n \quad\text{and}\quad |\mathcal{B}_{n,0}(\sigma) \cup \mathcal{B}_{n,1}(\sigma)| = nC_{n-1} = \binom{2n-2}{n-1}.$$

Sketch of proof. Let $\sigma \in \mathcal{S}_4$. The first identity is clear since $\pi \in \mathcal{B}_{n,0}(\sigma)$ if and only if $\pi \in \mathcal{S}_n(\sigma')$. For the second identity, we consider two cases:

(i) σ is of the form $4abc$ or $a4bc$, (ii) σ is of the form $ab4c$ or $abc4$.

In any case, $\sigma' = abc$. For $\pi \in \mathcal{S}_{n-1}(\sigma')$, let $\pi\lfloor k \rfloor_j \in \mathcal{S}_n(\sigma)$ be the expansion of π by k at position j.

Case (i): For $k \in [n]$, let

$$\hat{\pi}_k = \begin{cases} \pi\lfloor k \rfloor_1 & \text{if } \pi\lfloor k \rfloor_1 \in \mathcal{B}_{n,0}(\sigma) \cup \mathcal{B}_{n,1}(\sigma), \\ \pi\lfloor k \rfloor_m & \text{if } \pi\lfloor k \rfloor_1 \in \mathcal{B}_{n,m}(\sigma) \text{ with } m > 1. \end{cases}$$

If $\pi\lfloor k \rfloor_1$ contains the pattern σ', then k must be the first entry of any instance of σ'. Moreover, if $\pi\lfloor k \rfloor_1 \in \mathcal{B}_{n,m}(\sigma)$, then $\pi\lfloor k \rfloor_m \in \mathcal{B}_{n,1}(\sigma)$. Therefore, every $\pi \in \mathcal{S}_{n-1}(\sigma')$ generates n distinct permutations

$$\hat{\pi}_1, \dots, \hat{\pi}_n \in \mathcal{B}_{n,0}(\sigma) \cup \mathcal{B}_{n,1}(\sigma)$$

for a total of nC_{n-1} such permutations.

Conversely, if $\hat{\pi} \in \mathcal{B}_{n,0}(\sigma)$, then $\hat{\pi}$ avoids σ' and the permutation π obtained by removing the first entry from $\hat{\pi}$ gives an element of $\mathcal{S}_{n-1}(\sigma')$. On the other hand, if $\hat{\pi} \in \mathcal{B}_{n,1}(\sigma)$ has its only inactive site between $\hat{\pi}(i)$ and $\hat{\pi}(i+1)$, then we obtain a permutation $\pi \in \mathcal{S}_{n-1}(\sigma')$ by removing $\hat{\pi}(i)$ from $\hat{\pi}$. □

Group of 10

Maximal number of inactive sites $(n-2) \rightsquigarrow$ only 3 active.

Theorem. For every σ in the Wilf equivalence class of 1342 and $n \ge 3$, the set $\mathcal{B}_{n,n-2}(\sigma)$ is equinumerous with the set $\mathcal{S}^{\mathrm{ind}}_{n-1}(1342)$ of indecomposable permutations in $\mathcal{S}_{n-1}(1342)$ enumerated by A000257 in the OEIS:

1, 3, 12, 56, 288, 1584, 9152, 54912, 339456, ...

Sketch of proof. Let π' be the permutation obtained by removing n from $\pi \in \mathcal{S}_{n-1}(\sigma)$. For $\sigma \in \{2431, 2413, 3241\}$, the map

$$\pi \mapsto \pi' : \mathcal{B}_{n,n-2}(\sigma) \to \mathcal{S}^{\mathrm{ind}}_{n-1}(\sigma)$$

is bijective. Moreover, using the map $i \circ r \circ c$, we get a bijection

$$\mathcal{B}_{n,n-2}(4213) \to \mathcal{S}^{2\prec 1}_{n-1}(4132) \sim \mathcal{S}^{\mathrm{ind}}_{n-1}(4132).$$

Finally, $\mathcal{B}_{n,n-2}(4132) \sim \mathcal{B}_{n,n-2}(3142)$ follows from a result by Stankova [3]. The remaining patterns can be discussed using the reverse map since $\mathcal{B}_{n,n-2}(\sigma) = \mathcal{B}_{n,n-2}(r(\sigma))$. □

$2 \prec 1$ means having the '2' left from the '1'

Group of 12

Theorem. For every σ in the Wilf equivalence class of 3412,

$$|\mathcal{B}_{n+3,n+1}(\sigma)| = \sum_{k=0}^{n}\binom{n}{k} b_{k+1},$$

where $b_k = |\mathcal{S}_k(3412, 34\underline{21})|$. Appears to be A216947 in the OEIS:

1, 3, 11, 47, 225, 1173, 6529, 38265, 233795, ...

Assuming the below conjecture is true.

That the set $\mathcal{B}_{n,n-2}(\sigma)$ has the same number of elements for every $\sigma \in \{1234, 1243, 2134, 2143\}$ (and their reverses) follows from known connections between their generating trees (West [6, 7]).

For the four patterns in the symmetric class of 1432, we use different notions of active sites and prove that their enumeration gives the same sequences. Finally, $\mathcal{B}_{n,n-2}(1234) \sim \mathcal{B}_{n,n-2}(4123)$ follows from a result by Stankova [4].

An explicit combinatorial proof for the given formula will be provided in an upcoming preprint.

Conjecture. $\mathcal{S}_n(3412, 34\underline{21}) \sim S_n(2\underline{41}3, 3\underline{14}2)$.

General observations

- Wilf equivalent classes of pattern-avoiding permutations do not necessarily have isomorphic generating trees.
- In a permutation with maximal number of inactive sites relative to σ, the position of n is uniquely determined by σ.

Expansion operator

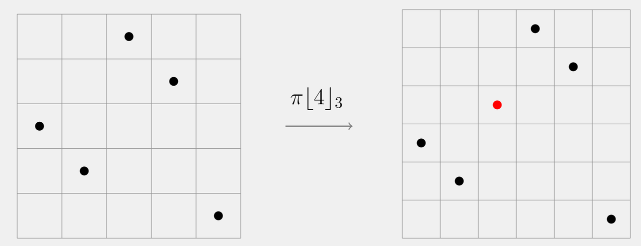
$\pi\lfloor k \rfloor_j$ is constructed from π as follows:

▷ increase every entry in π greater than or equal to k by one,

▷ move every entry at position greater than or equal to i one unit to the right, and

▷ insert k at position i.

For example, $(32541)\lfloor 4 \rfloor_3 = 324651$.



References

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Further conjectures

- The sequence defined by $s_n = |\mathcal{B}_{n,n-2}(4213) \cap \mathcal{S}_n(4132)|$ is the sequence of little Schröder numbers (A001003):

  1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, ...

- If $\mathcal{S}_n(\sigma) \sim \mathcal{S}_n(\tau)$, then $\mathcal{B}_{n,m}(\sigma) \sim \mathcal{B}_{n,m}(\tau)$. This is known to be true for certain classes in the group of 12.

Data for the equivalence class of 1234 (group of 12):

$n\backslash m$	0	1	2	3	4	5	6	7
3	5	1						
4	14	6	3					
5	42	28	22	11				
6	132	120	120	94	47			
7	429	495	585	577	450	225		
8	1430	2002	2695	3100	3021	2346	1173	
9	4862	8008	12012	15521	17491	16878	13058	6529

Data for the equivalence class of 1342 (group of 10):

$n\backslash m$	0	1	2	3	4	5	6	7
3	5	1						
4	14	6	3					
5	42	28	21	12				
6	132	120	112	92	56			
7	429	495	540	532	456	288		
8	1430	2002	2475	2760	2786	2448	1584	
9	4862	8008	11011	13530	15210	15568	13904	9152