Descents on quasi-Stirling permutations

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Descents and Eulerian polynomials

Definition

Let $\pi = \pi_1\pi_2\dots\pi_r$ be a sequence of positive integers.

i is a descent of π if $\pi_i > \pi_{i+1}$ or $i = r$.

$\operatorname{des}(\pi)$ = number of descents of π.

Example: $\operatorname{des}(36522131) = 5$

Let $\mathcal{S}_n$ = permutations of $\{1, 2, \dots, n\}$.

Definition (Eulerian polynomials)

$$A_n(t) = \sum_{\pi\in\mathcal{S}_n} t^{\operatorname{des}(\pi)}$$

It is well known that

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Stirling permutations

Definition (Gessel–Stanley '78)

A Stirling permutation is a permutation of the multiset $\{1,1,2,2,\ldots,n,n\}$ that avoids the pattern 212.

In other words, Stirling permutations $\pi_1\pi_2\ldots\pi_{2n}$ satisfy that, if $i<j<k$ and $\pi_i=\pi_k$, then $\pi_j>\pi_i$.

$\mathcal{Q}_n$ = set of Stirling permutations of $\{1,1,2,2,\ldots,n,n\}$.

Example

$$\mathcal{Q}_2=\{1122,1221,2211\}$$

Easy fact: $|\mathcal{Q}_n|=(2n-1)!!=(2n-1)\cdot(2n-3)\cdot\cdots\cdot3\cdot1.$

Stirling polynomials

Definition (Gessel–Stanley '78)

Stirling polynomials:

$$Q_n(t) = \sum_{\pi \in \mathcal{Q}_n} t^{\operatorname{des}(\pi)}$$

Example: $Q_3(t) = t + 8t^2 + 6t^3$

Definition

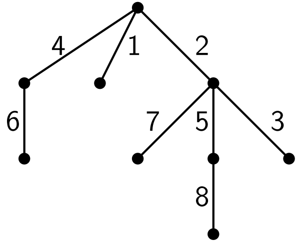
The Stirling number of the second kind $S(n,k)$ is the number of partitions of the set $\{1,2,\ldots,n\}$ into k blocks.

Theorem (Gessel–Stanley '78)

$$\sum_{m \geq 0} S(m+n, m)\, t^m = \frac{Q_n(t)}{(1-t)^{2n+1}},$$

Stirling permutations and trees

$\mathcal{I}_n$ = set of increasing edge-labeled plane rooted trees with n edges.



$\xrightarrow{\varphi}$ 4664112775885332

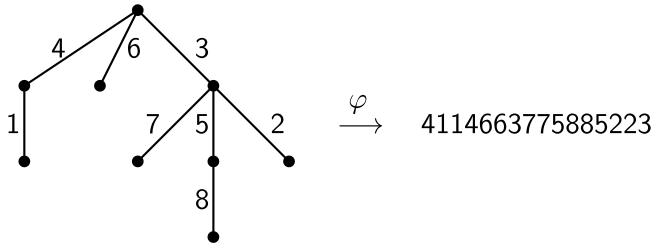
Theorem (Janson '08)

There is a bijection $\varphi : \mathcal{I}_n \longrightarrow \mathcal{Q}_n$ obtained by traversing the edges of the tree along depth-first walk from left to right, and recording their labels.

Let us now remove the increasing condition on the trees...

Quasi-Stirling permutations and trees

$\mathcal{T}_n$ = set of edge-labeled plane rooted trees with n edges.



Definition (Archer–Gregory–Pennington–Slayden '19)

A quasi-Stirling permutation is a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ that avoids the patterns 1212 and 2121.

In other words, it does not have four positions $i < j < k < \ell$ with $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$ (i.e., it is *non-crossing*).

Quasi-Stirling permutations

$\overline{\mathcal{Q}}_n$ = set of quasi-Stirling permutations of $\{1, 1, 2, 2, \ldots, n, n\}$.

Example

$$\overline{\mathcal{Q}}_2 = \{1122, 1221, 2211, 2112\}$$

Theorem (Archer–Gregory–Pennington–Slayden '19)

φ is a bijection between $\mathcal{T}_n$ and $\overline{\mathcal{Q}}_n$.

It follows that

$$|\overline{\mathcal{Q}}_n| = n!\,C_n = \frac{(2n)!}{(n+1)!}.$$

Our first result settles a conjecture of the above authors:

Theorem

The number of $\pi \in \overline{\mathcal{Q}}_n$ with $\operatorname{des}(\pi) = n$ is equal to $(n+1)^{n-1}$.

Descents on quasi-Stirling permutations

More generally, we are interested in the distribution of des on $\overline{\mathcal{Q}}_n$.

Define the quasi-Stirling polynomials

$$\overline{Q}_n(t) = \sum_{\pi \in \overline{\mathcal{Q}}_n} t^{\mathrm{des}(\pi)}.$$

Example: $\overline{Q}_3(t) = t + 13t^2 + 16t^3$

We will give an expression for their exponential generating function (EGF):

$$\overline{Q}(t,z) = \sum_{n \geq 0} \overline{Q}_n(t) \frac{z^n}{n!}.$$

But first, recall the well-known EGF for the Eulerian polynomials:

$$A(t,z) = \sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{1-t}{1-te^{(1-t)z}}.$$

Descents on quasi-Stirling permutations

Theorem (Main theorem)

The EGF $\overline{Q}(t,z)$ *for quasi-Stirling permutations by the number of descents satisfies the implicit equation*

$$\overline{Q}(t,z) = A(t, z\overline{Q}(t,z)),$$

that is,

$$\overline{Q}(t,z) = \frac{1-t}{1-te^{(1-t)z\overline{Q}(t,z)}}.$$

Its coefficients satisfy

$$\overline{Q}_n(t) = \frac{n!}{n+1}\,[z^n]A(t,z)^{n+1}.$$

Here $[z^n]F(z)$ denotes the coefficient of z^n in $F(z)$.

Consequences

In analogy to

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}} \qquad \text{(Eulerian)}$$

$$\sum_{m\geq 0} S(m+n,m)\, t^m = \frac{Q_n(t)}{(1-t)^{2n+1}} \qquad \text{(Stirling)}$$

we get

Theorem

$$\sum_{m\geq 0} \frac{m^n}{n+1}\binom{m+n}{m}\, t^m = \frac{\overline{Q}_n(t)}{(1-t)^{2n+1}} \qquad \textit{(quasi-Stirling)}$$

Open: Find a combinatorial proof.

Properties of quasi-Stirling polynomials

Definition: i is a plateau of π if $\pi_i = \pi_{i+1}$,
i is an ascent of π if $\pi_i < \pi_{i+1}$ or $i = 0$.

Theorem (Bóna '08)

On average, Stirling permutations in $\mathcal{Q}_n$ have $(2n+1)/3$ ascents, $(2n+1)/3$ descents, and $(2n+1)/3$ plateaus.

Theorem

On average, quasi-Stirling permutations in $\overline{\mathcal{Q}}_n$ have $(3n+1)/4$ ascents, $(3n+1)/4$ descents, and $(n+1)/2$ plateaus.

Properties of quasi-Stirling polynomials

Theorem (Frobenius)

The roots of the Eulerian polynomials $A_n(t)$ are real, distinct, and nonpositive.

Theorem (Brenti'89, Bóna'08)

The same holds for the Stirling polynomials $Q_n(t)$.

Theorem

The same holds for the quasi-Stirling polynomials $\overline{Q}_n(t)$.

Corollary

- *The coefficients of $\overline{Q}_n(t)$ are unimodal and log-concave.*
- *The distribution of the number of descents on $\overline{\mathcal{Q}}_n$ converges to a normal distribution as $n \to \infty$.*

k-Stirling and k-quasi-Stirling permutations

Gessel and Stanley proposed a generalization of Stirling permutations:

Definition (Gessel–Stanley '78)

A k-Stirling permutation is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ that avoids the pattern 212.

These are in bijection with certain decorated increasing trees.

It is natural to define the following, which correspond to same trees without the increasing condition:

Definition

A k-quasi-Stirling permutation is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ that avoids the patterns 1212 and 2121.

We have generalized our main theorem to give an implicit formula for the EGF of k-quasi-Stirling permutations with respect to the number of ascents, descents and plateaus.