Pattern classes equinumerous to the class of ternary forests

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Joint work with Megan Martinez

2020 Permutation Patterns Virtual Workshop
June 30 – July 1, 2020

The sequence

- Sequence A098746:

$$1, 1, 2, 6, 23, 102, 495, 2549, 13682, 75714, \ldots$$

- $$a_n = \sum_{k=0}^{n} \frac{n-k}{2k+n} \binom{2k+n}{k}, \quad n \geqslant 0$$

- Enumerates: forests of planted ternary trees
- Generating function:

$$F = F(x) = \frac{1}{1 - xT},$$

where

$$T = 1 + xT^3 = C(xT), \quad C = \text{Catalan o.g.f.}$$

A098746 in pattern avoiding permutations

- Albert et al. (2004): A098746 enumerates $Av(4231, 42513)$
  - $Av(4231)$ – loosely locked jump queue (a $\mathcal{C}(231)$ C-machine)
  - $Av(4231, 42513)$ – strictly locked jump queue

A098746 in pattern avoiding inversion sequences

An inversion sequence $e = e_1 \ldots e_n \in I_n = \prod_{i=0}^{n} [0, i-1]$ is an integer string such that $0 \leqslant e_i \leqslant n-1$ for all $i = 1, 2, \ldots, n$.

- A098746 enumerates $\mathbf{I}(100, 201, 210)$, $\mathbf{I}(101, 201, 210)$, and $\mathbf{I}(110, 201, 210)$
  - conjectured – Martinez, Savage (2018),
  - proved for $\mathbf{I}(100, 201, 210)$ – Martinez (2018), B. (2020)
- $\mathbf{I}(100, 201, 210) \cong \mathbf{I}(110, 201, 210)$ – Martinez, Savage (2018)
  - preserves the set of distinct letters in an inversion sequence
- $\mathbf{I}(100, 201, 210) \cong \mathbf{I}(101, 201, 210)$ – B. (2020)
  - preserves the set of distinct letters in an inversion sequence

Wilf class enumerated by A098746

Contains 16 $(4, 5)$-symmetry classes of pairs of permutation patterns and 3 triples of inversion sequence patterns.

Permutations	
1243	21453
1324	12543
1324	14532
1324	15342
1324	24153
1342	13254
1342	15243
1342	15324
1342	15423
1342	24351
1342	31542
1342	34512
1342	41253
2143	12543
2413	15324

Inversion seq.		
100	201	210
101	201	210
110	201	210

Bijection from $\mathrm{I}(100, 201, 210)$ to lattice paths

Straightforward to verify:

A098746(n) enumerates lattice paths $(0, 0) \rightsquigarrow (n, n)$ on or below the diagonal with unit steps $E = (1, 0)$, $N_1 = (0, \frac{1}{2})$, $N_2 = (0, 1)$, such that

- each N_1 is followed by E or ends the path, and
- no valley ($N_i E$ block, $i = 1, 2$) has the “corner” at $(k, k - \frac{1}{2})$, for any $1 \leqslant k \leqslant n$.

Bijection from $\mathrm{I}(100, 201, 210)$ to lattice paths

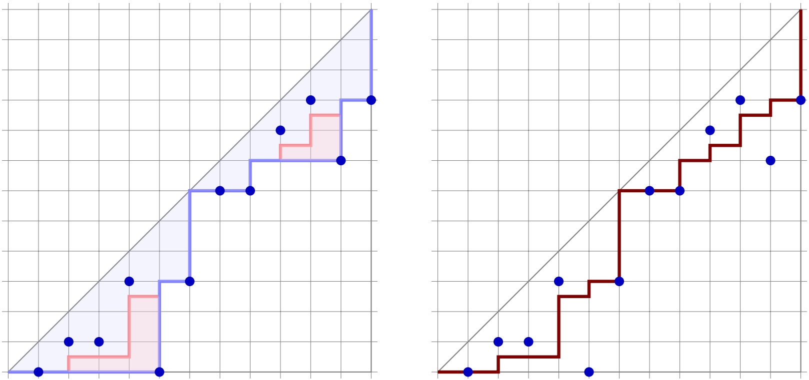


Image of $e = 011303668979 \in \mathrm{I}_{12}(100, 201, 210)$

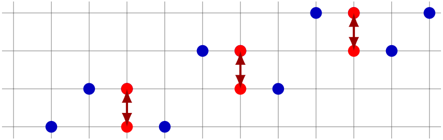
Bijection between $\mathrm{I}(100, 201, 210)$ and $\mathrm{I}(110, 201, 210)$

Martinez, Savage (2018):

- $\Phi : \mathrm{I}(100, 201, 210) \to \mathrm{I}(110, 201, 210)$: given $e = e_1 \dots e_n \in \mathrm{I}_n(100, 201, 210)$, for each $i = 1, 2, \dots, n$, in that order,
  - check if e_i is the second 1 in an instance of pattern 110 in e.
  - if it is, change e_i so as to turn this instance of pattern 110 into an instance of pattern 100 on the same letters. Otherwise, leave e_i unchanged.
- $\Phi^{-1} : \mathrm{I}(110, 201, 210) \to \mathrm{I}(100, 201, 210)$: given $e = e_1 \dots e_n \in \mathrm{I}_n(110, 201, 210)$, for each $i = n, n-1, \dots, 1$, in that order,
  - check if e_i is the first 0 in an instance of pattern 100 in e,
  - if it is, find the maximal 1 among such instances of 100 and change e_i so as to turn that instance of pattern 100 into an instance of pattern 110 on the same letters. Otherwise, leave e_i unchanged.

$\mathrm{I}(100, 201, 210) \overset{\cong}{\longleftrightarrow} \mathrm{I}(110, 201, 210)$ – an example

$\mathrm{I}(100,201,210) \ni 0110221332 3 \longleftrightarrow 0100211322 3 \in \mathrm{I}(110,201,210)$



Φ : scan $\rightarrow$, change $\downarrow$

Φ^{-1} : scan $\leftarrow$, change $\uparrow$

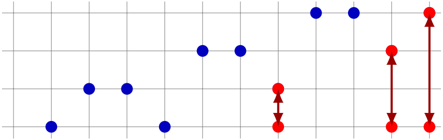
Bijection between $\mathrm{I}(100, 201, 210)$ and $\mathrm{I}(101, 201, 210)$

B. (2020):

- $\Psi : \mathrm{I}(100, 201, 210) \to \mathrm{I}(101, 201, 210)$: given $e = e_1 \ldots e_n \in \mathrm{I}_n(101, 201, 210)$, for each $i = 1, 2, \ldots, n$, in that order,
  - check if e_i is the second 1 in an instance of pattern 101 in e.
  - if it is, change e_i so as to turn this instance of pattern 101 into an instance of pattern 100 on the same letters. Otherwise, leave e_i unchanged.
- $\Psi^{-1} : \mathrm{I}(101, 201, 210) \to \mathrm{I}(100, 201, 210)$: given $e = e_1 \ldots e_n \in \mathrm{I}_n(101, 201, 210)$, for each $i = n, n-1, \ldots, 1$, in that order,
  - check if e_i is the second 0 in an instance of pattern 100 in e,
  - if it is, find the maximal 1 among such instances of 100 and change e_i so as to turn that instance of pattern 100 into an instance of pattern 101 on the same letters. Otherwise, leave e_i unchanged.

$\mathrm{I}(100, 201, 210) \stackrel{\cong}{\longleftrightarrow} \mathrm{I}(101, 201, 210)$ – an example

$$\mathrm{I}(100, 201, 210) \ni 0110221\color{red}{3}\color{black}{3}\color{red}{23} \longleftrightarrow 0110220\color{red}{}33\color{red}{00}\color{black} \in \mathrm{I}(101, 201, 210)$$



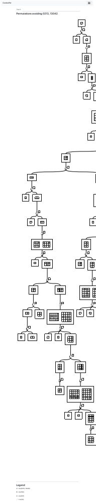
Ψ : scan $\rightarrow$, change $\downarrow$

Ψ^{-1} : scan $\leftarrow$, change $\uparrow$

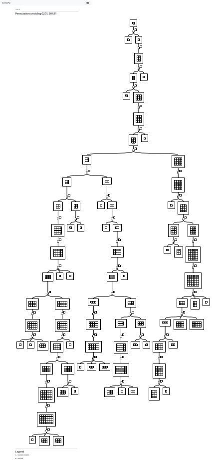
Back to permutations – Tilescope structure trees

Albert, Bean, Claesson, Pantone, Úlfarsson – TileScope for Combinatorial Exploration

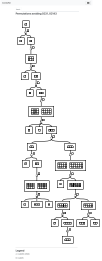
$Av(4231, 42513)$



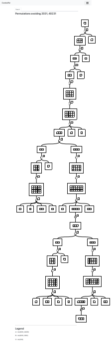
$Av(4132, 41523)$



$Av(4132, 53412)$



$Av(3142, 51342)$



After 90° ccw rotation:

$Av(3142, 14352)$

Statistics on avoidance sets

Pantone, 2019: Many standard statistics (singletons, pairs, triples) appear to be jointly equidistributed on several pattern sets: e.g. number/positions/values of descents, descent runs, LRmin, LRmax, RLmin, RLmax, first entry, last entry, number/sizes of $\oplus$-decomposition, etc. – “the usual suspects”.

Statistics on avoidance sets – an example

Conjecture (Pantone, 2019)

The following statistics are jointly equidistributed on $\mathcal{A}v(4132, 53412)$ *and* $\mathcal{A}v(3142, 14352)$*:*

- *number of descents,*
- *values of RLmin (hence, number of RLmin and value of last entry),*
- *number of decreasing runs,*
- *length of longest descending run.*

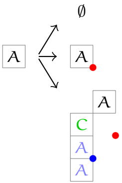
Theorem (B., 2019)

This holds at least for

- *number of descents,*
- *values of RLmin (hence, number of RLmin and value of last entry).*

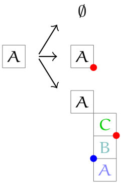
Proof – initial decomposition

$Av(4132, 53412)$



$A = Av(4132, 53412)$
$C = Av(312)$

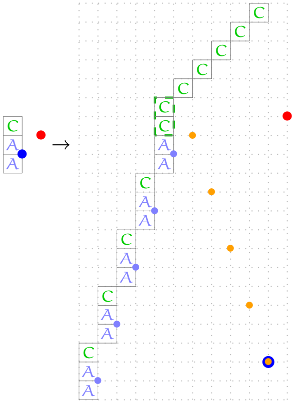
$Av(3142, 14352)$



$A = Av(3142, 14352)$
$B = Av(3142, 4231)$
$C = Av(213)$

Proof – decomposition of the xT block

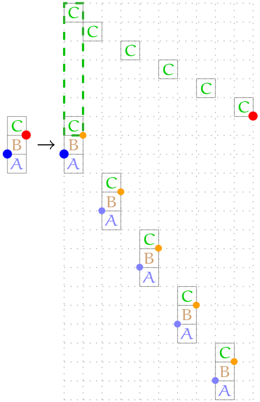
$Av(4132, 53412)$



$A = Av(4132, 53412)$
$C = Av(312)$ (dashbox CC also avoids 312)

l.h.s. blocks are possibly just "●"
nonempty r.h.s. subblocks are possibly just "●"

$Av(3142, 14352)$



$A = Av(3142, 14352)$
$B = Av(3142, 4231)$
$C = Av(213)$ (dashbox CC also avoids 213)

Statistics on permutation classes and inversion sequence classes

Martinez, Savage (2018): dis, the number of distinct letters statistic, is equidistributed on $\mathbf{I}(100, 201, 210)$ and $\mathbf{I}(110, 201, 210)$.

B. (2020): Same for dis on $\mathbf{I}(100, 201, 210)$ and $\mathbf{I}(101, 201, 210)$.

Conjecture

The statistic $1 + \mathrm{asc}$ *(number of ascents plus 1) on the classes*

$$\begin{aligned} &Av(4231, 42513), \\ &Av(4132, 53412), \\ &Av(4132, 41523), \\ &Av(3241, 34152), \\ &Av(3142, 51342), \end{aligned}$$

has the same distribution as dis*, the number of distinct letters, on*

$$\mathbf{I}(100, 201, 210), \qquad \mathbf{I}(110, 201, 210), \qquad \mathbf{I}(101, 201, 210).$$

Plus-indecomposables

Easy to see that the $\oplus$-indecomposable blocks in each of the 5 permutation classes and 3 inversion classes above are counted by the ternary numbers (i.e. have xT as the generating function).

Theorem

In each of the classes

$$\mathbf{I}(100, 201, 210), \quad \mathbf{I}(101, 201, 210), \quad \mathbf{I}(110, 201, 210),$$

- *the 00-starting inversion sequences (together with* 0*) are counted by the ternary numbers (have generating function* xT*).*
- dis *has the same distribution on* $\oplus$*-indecomposables and on 00-starting sequences (together with* 0*) in each of the three classes.*
- *the distribution of* dis *on these subsets is given by A091320 (also counts number of leaves in noncrossing trees).*

More statistic distributions on $\oplus$-indecomposables

stat	$I_n(100, 201, 210)$	$I_n(110, 201, 210)$	$I_n(101, 201, 210)$
$\mathtt{dis}$	A091320	A091320	A091320
$\mathtt{inv}$	A108410	A108410	
$\mathtt{rlmin}$	A120986	A120986	
$\mathtt{lrmax}$	same distribution, not in OEIS		
$\mathtt{des}$	same distribution, not in OEIS		

$\mathtt{dis}$ = # distinct letters

$\mathtt{inv}$ = # inversions

$\mathtt{rlmin}$ = # right-to-left minima

$\mathtt{lrmax}$ = # left-to-right maxima

$\mathtt{des}$ = # descents

Further questions for pattern classes in this Wilf class

- Corresponding statistics on the permutation pattern classes in the same Wilf class?
- (Equi)distributions of other statistics on pattern classes in this Wilf class?
- A bijection between from permutation pattern classes and inversion sequence classes or other ternary forest related objects?

References

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