Distributions of mesh patterns of short lengths

Philip Biao Zhang

Tianjin Normal University

joint with Sergey Kitaev

2020 Permutation Patterns Virtual Workshop

Mesh patterns

The notion of a mesh pattern, generalizing several classes of patterns, was introduced by Brändén and Claesson to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns.

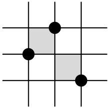
A pair (τ, R), where τ is a permutation of length k and R is a subset of $[\![0,k]\!] \times [\![0,k]\!]$, where $[\![0,k]\!]$ denotes the interval of the integers from 0 to k, is a mesh pattern of length k.

- P. Brändén and A. Claesson, Mesh patterns and the expansion of permutation statistics as sums of permutation patterns, *Electronic J. Combin.*, 18(2) (2011), 5.

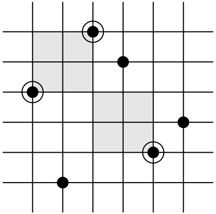
Let (i, j) denote the box whose corners have coordinates $(i, j), (i, j + 1), (i + 1, j + 1)$, and $(i + 1, j)$.

Let the horizontal lines represent the values, and the vertical lines denote the positions in the pattern.

Mesh patterns can be drawn by shading the boxes in R. The following picture represents the mesh pattern with $\tau = 231$ and $R = \{(1, 2), (2, 1)\}$:



For instance, the permutation 416524 contains the above mesh pattern:



The first systematic study of mesh patterns was not done until

- I. Hilmarsson, I. Jónsdóttir, S. Sigurdardóttir, L. Vidarsdóttir, and H. Ulfarsson, Wilf-classification of mesh patterns of short length, *Electr. J. Combin.*, 22(4) (2015), 13.

where 25 out of 65 non-equivalent **avoidance** cases of patterns of length 2 were solved. That is, in the 25 cases, the number of permutations avoiding the respective mesh patterns was found.

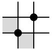
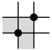
In our research, we initiate a systematic study of **distributions** of mesh patterns by giving 27 distribution results for the patterns considered by Hilmarsson et al. including 14 distributions for which avoidance was not known.

Moreover, for the unsolved cases, we prove an equidistribution result (out of 6 equidistribution results we prove in total), and conjecture 6 more equidistributions.

Techniques used by us include generating functions, recurrence relations, and bijections. The results in this presentation are from

- S. Kitaev and P. B. Zhang, Distributions of mesh patterns of short lengths, *Adv. in Appl. Math.*, 110 (2019), 1–32.

We note that from the distribution point of view, we cannot consider just the 65 patterns presented by Hilmarsson et al., since there are more patterns to consider.

For example, the pattern Nr. 39 =  was considered there, while its Wilf-equivalent pattern (by the *Shading Lemma*)  was not considered.

However, these two patterns have different distributions.

Two patterns, p_1 and p_2, are said to be Wilf-equivalent if for any $n \geq 0$, the number of permutations of length n avoiding p_1 is equal to that avoiding p_2.

Let S_n be the set of all permutations of length n, which we call n-permutations.

For example, $S_3 = \{123, 132, 213, 231, 312, 321\}$.

For a pattern p and a permutation π, we let $p(\pi)$ denote the number of occurrences of p in π.

Also, let $S_n(p)$ denote the set of all permutations of length n avoiding p and $S(p) = \cup_{n\geq 0} S_n(p)$.

Finally, let

$$F(x) = \sum_{n\geq 0} n! x^n.$$

Our main enumerative method is via deriving a functional equation for the generating function in question, and solving it; we use recurrence relations in the remaining cases.

Occurrences of the pattern  are known as strong fixed points. However, the distribution of strong fixed points seems to be unknown before our paper.

Theorem

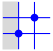
Let

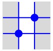
$F(x,q) = \sum_{n\geq 0} x^n \sum_{\pi \in S_n} q$  $(\pi) = \sum_{n\geq 0} x^n \sum_{\pi \in S_n} q$  (π),

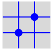
and $A(x)$ be the g.f. for $S($  *$) = S($*  *$)$. Then,*

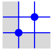
$$A(x) = \frac{F(x)}{1 + xF(x)}; \qquad F(x,q) = \frac{F(x)}{1 + x(1-q)F(x)}.$$

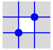
By a "trivial" distribution we mean the situation when either the pattern in question can occur at most once and its avoidance was given by Hilmarsson et al. or pattern's occurrences can easily be understood from the shape of the pattern. There are 10 such patterns:

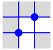
Nr. 5 = 

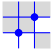
Nr. 10 = 

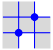
Nr. 11 = 

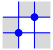
Nr. 12 = 

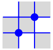
Nr. 13 = 

Nr. 18 = 

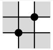
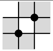
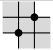
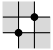
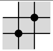
Nr. 19 = 

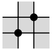
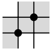
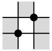
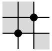
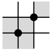
Nr. 20 = 

Nr. 21 = 

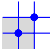
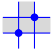
Nr. 22 = 

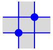
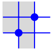
We use the approach similar to, but in several cases (much) more involved than, the proof of Theorem 1.1 to find the distribution and, whenever appropriate, avoidance for the following 12 patterns:

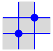
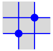
Nr.	Repr. p	$A(x)$	$F(x,q)$
16		$\frac{(1+x)F(x)}{1+xF(x)}$	$\sum_{i\geq 0} q^{\binom{i}{2}} x^i \prod_{j=0}^{i} \frac{F(q^j x)}{1+q^j xF(q^j x)}$
17		$\left(1-x+\frac{x}{1+xF(x)}\right)F(x)$	$\left(1-x+\frac{x}{1+(1-q)xF(x)}\right)F(x)$
27		$F(x)-\frac{x^2F(x)^3}{1+xF(x)}$	$F(x)-\frac{(1-q)x^2F^3(x)}{1+(1-q)xF(x)}$
28		$\frac{F(x)}{1+x^2F^2(x)}$	$\frac{F(x)}{1+(1-q)x^2F^2(x)}$
30		$\frac{(1+x)F(x)}{1+x+x^2F(x)}$	$\frac{(1+x-qx)F(x)}{1+(1-q)x+(1-q)x^2F(x)}$
33		$\frac{(1+2xF(x))F(x)}{(1+xF(x))^2}$	$\sum_{i=0}^{\infty} q^{\binom{i}{2}} x^i \left(\frac{F(x)}{1+xF(x)}\right)^{i+1}$
34		$\frac{F(x)}{1+x^2F(x)}$	$\frac{F(x)}{1+(1-q)x^2F(x)}$

Nr.	Repr. p	$A(x)$	$F(x,q)$
55		$\frac{F(x)}{1+x(F(x)-1)}$	$\frac{F(x)}{1+(1-q)x(F(x)-1)}$
56			
63		$\frac{2F(x)-1}{F(x)}$	$\frac{(2-q)F(x)+q-1}{(1-q)F(x)+q}$
64			
65			

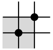
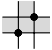
Next, we consider six patterns for which our generating functions approach does not work. Instead, we derive recurrence relations for the distribution of respective patterns. The patterns are:

Nr. 8 = 
Nr. 9 = 

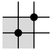
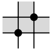
Nr. 14 = 
Nr. 15 = 

Nr. 36 = 
Nr. 45 = 

In what follows, we denote by $T_{n,k}$ the number of n-permutations with k occurrences of the pattern in question. Also, let $T_n(x) = \sum_{k=0}^{n-1} T_{n,k}x^k$.

The distributions for the patterns Nr. 8 =  and Nr. 9 =  are given by the unsigned Stirling numbers of the first kind (the sequence A132393 in the OEIS):

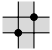
Theorem (Nr. 8 and Nr. 9)

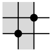
Both patterns p_1 =  *and* p_2 =  *satisfy*

$$T_{n,k} = T_{n-1,k-1} + (n-1)\,T_{n-1,k} \tag{1}$$

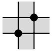
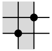
with the initial conditions $T_{n,0} = (n-1)!$ *for* $n \geq 1$ *and* $T_{0,0} = 1$*, which shows that* $T_{n,k} = C(n, k+1)$*, the unsigned Stirling number of the first kind. The row generating function for* $T_{n,k}$ *is given by*

$$\sum_{k=0}^{n-1} T_{n,k} x^k = \prod_{i=1}^{n-1} (x+i). \tag{2}$$

An occurrence of the pattern Nr. 14 =  is known as a small ascent, and its reverse as a small descent. The distribution of this pattern is given by the sequence A123513.

The next theorem derives a recurrence relation for the pattern and shows that the same recurrence relation works for the pattern Nr. 15 =  thus establishing equidistribution of these patterns.

Theorem (Nr. 14 and Nr. 15)

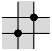
Both patterns $p_1 =$  *and* $p_2 =$  *satisfy the recurrence relation*

$$T_{n,k} = T_{n-1,k-1} + (k+1)\,T_{n-1,k+1} + (n-k-1)\,T_{n-1,k} \tag{3}$$

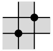
with the initial conditions $T_{1,0} = 1$, $T_{2,0} = 1$, $T_{2,1} = 1$. *Equivalently,*

$$T_n(x) = (x+n-1)\,T_{n-1}(x) + (1-x)\,T'_{n-1}(x) \tag{4}$$

with the initial conditions $T_1(x) = 1$ *and* $T_2(x) = 1 + x$.

Next, we find the recurrence relation for the distribution of the pattern Nr. 36 = . Note that an occurrence of the pattern is a small ascent in which the left element is a left-to-right minimum, that is, an element having no smaller elements to the left of it.

Theorem (Nr. 36)

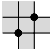
The pattern $p =$  *satisfies the recurrence relation*

$$T_{n,k} = (k+1)\,T_{n-1,k+1} + (n-k)\,T_{n-1,k} - T_{n-2,k} + T_{n-2,k-1} \tag{5}$$

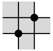
with the initial conditions $T_{1,0} = 1$, $T_{2,0} = 1$, $T_{2,1} = 1$. *Equivalently,*

$$T_n(x) = n\,T_{n-1}(x) + (1-x)\,T'_{n-1}(x) + (x-1)\,T_{n-2}(x) \tag{6}$$

with the initial conditions $T_1(x) = 1$ *and* $T_2(x) = 1 + x$.

Now, we find the recurrence relation for the distribution of the pattern Nr. 45 = , which is the most difficult case among the recurrence relations.

Theorem (Nr. 45)

The pattern $p =$  *satisfies the recurrence relation*

$$\begin{aligned} T_{n,k} = & (k+1)\,T_{n-1,k+1} + (n-k-1)\,T_{n-1,k} + T_{n-1,k-1} \\ & + (k+1)\,T_{n-2,k+1} + (n-2k-2)\,T_{n-2,k} - (n-k-1)\,T_{n-2,k-1} \end{aligned} \tag{7}$$

with the initial conditions $T_{1,0} = 1$, $T_{2,0} = 1$, $T_{2,1} = 1$. *Equivalently,*

$$\begin{aligned} T_n(x) = & (x+n-1)\,T_{n-1}(x) + (1-x)\,T'_{n-1}(x) \\ & + (n-2)(1-x)\,T_{n-2}(x) + (1-x)^2\,T'_{n-2}(x), \end{aligned} \tag{8}$$

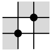
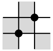
with the initial conditions $T_1(x) = 1$ *and* $T_2(x) = 1 + x$.

We were not able to find the distributions of the folllwing three patterns.

Conjecture

The patterns Nr. 48 = , *Nr. 49 =* , *Nr. 50 =*  *have the same distribution.*

Theorem (Nr. 48 and Nr. 49)

The patterns $p_1 =$  *and* $p_2 =$  *are equidistributed.* **Structures in question:**

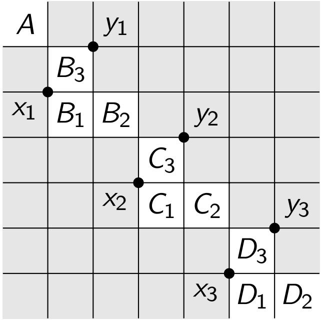


Fig 1: a permutation π with three occurrences of the pattern p_1

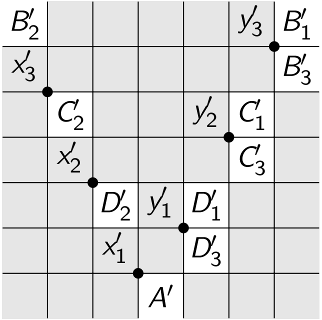
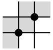
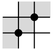
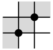


Fig 2: a permutation π with three occurrences of the pattern p_2

1. Conjectured equidistributions of Nr. 48 (49) and 50

The structure of permutations with k occurrences of the pattern Nr. 50 =  is as given in Fig. 3 for $k = 3$, where X_1 and X_3 can be any permutations, and X_2 and A must be  - avoiding for $X \in \{B, C, D\}$.

Even though the structure in Fig. 3 is very similar to those in Fig. 1 and 2 corresponding to the patterns Nr. 48 and Nr. 49, respectively, we were not able to find a bijective proof showing the conjectured equidistribution of the three patterns.

Indeed, there is a problem with  - avoiding blocks X_2 in Fig. 3, for $X \in \{B, C, D\}$, having relations with both X_1 and X_3, horizontally and vertically, while in Fig. 1 and 2 the respective blocks X_2 have only horizontal relations with X_1, and X_1 and X_3 have vertical relations.

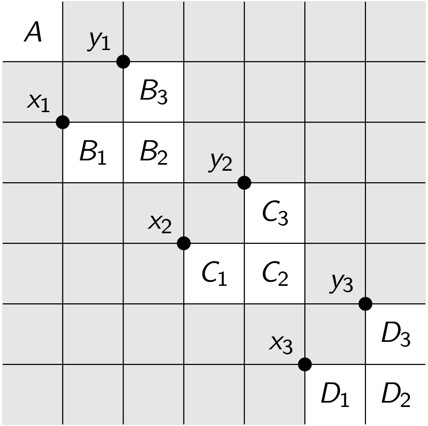
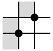


Fig 3: Related to Pattern Nr. 50

2. Conjectured distribution for Nr. 3

Conjecture

The distribution of the pattern Nr. 3 =  is given by the sequence A200545 in the OEIS, which is the triangle, read by rows, given by

$$(1, 0, 2, 1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6, \ldots)\ DELTA\ (0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots)$$

where DELTA is the operator defined in *A084938* in terms of continued fraction: the triangle $[r_0, r_1, \ldots]$ *DELTA* $[s_0, s_1, \ldots]$ has generating function

$$\frac{1}{1 - \frac{r_0x+s_0xy}{1-\frac{r_1x+s_1xy}{1-\frac{r_2x+s_2xy}{1-\ldots}}}}.$$

Note that the operator DELTA was already linked to patterns in permutations, and also to so-called Riordan arrays, in A200545 (OEIS).

3. Joint distribution of patterns

As a direction for further research, we suggest studying joint distribution of patterns considered in this paper and other permutation statistics.

As an illustration of this idea, we derive the following generating function

$$F(x,q,t)=\sum_{n\geq 0}x^n\sum_{\pi\in S_n}q^{\boxed{\cdot}(\pi)}\,t^{\mathrm{des}(\pi)}$$

generalizing our first theorem above.

4. Final remarks

As a final remark, we note that it would be interesting to classify completely mesh patterns of length 2 with respect to their distribution.

As noted above, the number of equivalence classes here is larger than that of Wilf-equivalence classes (given by equivalence with respect to avoidance) discussed in Hilmarsson et al.

Also, note the appearance of the following follow up paper to our results.

- S. Kitaev, P. B. Zhang and X. Zhang. Distributions of several infinite families of mesh patterns, *Appl. Math. Comput.*, 372 (2020), 124984.