On partially ordered patterns of length 4 and 5 in permutations

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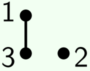
Partially ordered patterns

Definition

A partially ordered pattern (POP) p of length k is defined by a k-element partially ordered set (poset) P labeled by the elements in $\{1,\ldots,k\}$. An occurrence of such a POP p in a permutation $\pi = \pi_1 \cdots \pi_n$ is a subsequence $\pi_{i_1} \cdots \pi_{i_k}$, where $1 \le i_1 < \cdots < i_k \le n$, such that $\pi_{i_j} < \pi_{i_m}$ if and only if $j < m$ in P.

Examples

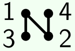
Any classical pattern of length k corresponds to a k-element chain.

The POP $p =$ 

occurs five times in the permutation 41523, namely, as the subsequences 412, 413, 452, 453, and 523. Clearly, avoiding p is the same as avoiding the patterns 312, 321 and 231 at the same time.

Partially ordered patterns

One-line notation

POPs can also be defined using one-line notation by providing the **minimal set of relations** defining the respective poset. For example, the POP in the example above can be defined by $\{1 > 3\}$, while the POP $p =$ 

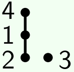
can be defined by $\{1 > 3, 1 > 2, 4 > 2\}$.

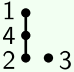
Significance of POPs

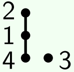
POPs provide a uniform notation for several combinatorial structures such as peaks, valleys, modified maxima and minima, p-descents in permutations, and others; see [S. Kitaev. A survey on partially ordered patterns. In Permutation Patterns (2010), Linton, Ruskuc, and Vatter, Eds., vol. 376 of London Math. Soc. Lect. Note Ser., Cambridge University Press, 115–135.]. Also, POPs provide a convenient language to deal with larger sets of permutation patterns.

Partially ordered patterns

POPs are natural in the theory of permutation patterns

For example, the simultaneous avoidance of the patterns 3214, 3124, 2134, and 2143 considered, up to trivial bijections, in [Defant. Stack-sorting preimages of permutation classes, 2018, https://arxiv.org/abs/1809.03123.] is nothing else but the avoidance of the POP 

, which suggests natural directions of research to study the avoidance of the POPs 

, 

, etc.

POPs in the literature

POPs were studied in the context of permutations, words and compositions in the literature.

Our research project

Original goal of the project

To explore **exhaustively** the variety of objects in the Online Encyclopedia of Integer Sequences (OEIS) that are **equinumerous to length 4, 5 POP-avoiding permutations**, and to justify any observations, which often require **non-trivial enumeration** or a **bijection**, but sometimes are given "for free" via known pattern-avoidance studied in different terms. Some of our results for length 4, 5 POP-avoiding permutations follow from **more general theorems** we prove.

A key tool in our studies

The software produced by Stephen Gardiner in 2018 as part of his MSc studies at the University of Strathclyde is able to go **exhaustively** through all POPs of length 4 and 5 (length 3 POPs are rather trivial and were omitted) and detect any connections to the OEIS.

An overview of our results

“High level description”

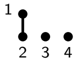
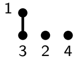
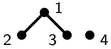
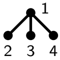
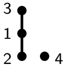
Observing connections to **38 sequences** in the OEIS, out of which **18 sequences** have no known interpretation in terms of pattern avoidance. We justified **all but 6** connections all related to POPs of length 5. Also, in our studies, we obtain **13 new enumerations** for pattern avoiding permutations for patterns of length 4 and 5, in particular, contributing to a long line of enumerative results on length 4 permutation patterns, e.g. with enumeration of triples of such patterns being concluded in 2017 by Callan, Mansour and Shattuck.

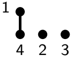
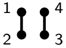
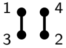
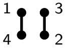
The paper produced

A. L.L. Gao, S. Kitaev. On partially ordered patterns of length 4 and 5 in permutations, *Elect. J. Comb.* **26** (2019) 3, 31pp.

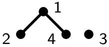
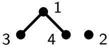
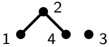
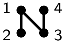
The results coming from our general results

In the tables below, for the highlighted OEIS sequences, **no interpretation** in terms of **permutation patterns** was known until our work.

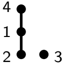
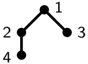
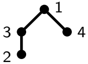
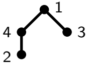
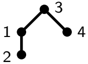
POP	Sequence (beginning with $n = 1$)	OEIS
	1, 2, 6, 12, 20, 30, 42, 56, 72, ...	A103505
	1, 2, 6, 12, 25, 48, 91, 168, 306, ...	A045925
	1, 2, 6, 16, 40, 96, 224, 512, 1152, ...	A129952 A057711
	1, 2, 6, 18, 54, 162, 486, 1458, 4374,...	A025192
	1, 2, 6, 20, 68, 232, 792, 2704, 9232, ...	A006012
	1, 2, 6, 20, 70, 252, 924, 3432, 12870,...	A000984

POP	Sequence (beginning with $n = 1$)	OEIS
	1, 2, 6, 12, 25, 57, 124, 268, 588,...	A214663 A232164
	1, 2, 6, 18, 50, 130, 322, 770, 1794,...	A048495
	1, 2, 6, 18, 52, 152, 444, 1296, 3784,...	A077835
	1, 2, 6, 18, 50, 134, 358, 962, 2594,...	A271897

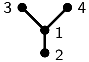
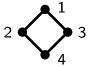
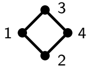
More POPs of length 4 with longest chain of size 2

POP	Sequence (beginning with $n = 1$)	OEIS
	1, 2, 6, 16, 40, 100, 252, 636, 1604,...	A111281
	1, 2, 6, 16, 44, 120, 328, 896, 2448,...	A002605
	1, 2, 6, 16, 42, 110, 288, 754, 1974,...	A111282
	1, 2, 6, 19, 59, 180, 544, 1637, 4917,...	A111277
	1, 2, 6, 19, 60, 189, 595, 1873, 5896,...	A052544 A204200

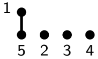
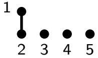
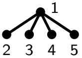
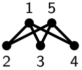
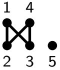
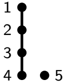
POPs of length 4 with longest chain of size 3

POP	Sequence (beginning with $n = 1$)	OEIS
	1, 2, 6, 20, 71, 264, 1015, 4002, 16094,...	A049124
	1, 2, 6, 21, 80, 322, 1346, 5783, 25372,...	A257561
	1, 2, 6, 21, 79, 309, 1237, 5026, 20626,...	A111279
	1, 2, 6, 21, 80, 322, 1347, 5798, 25512,...	A106228
	1, 2, 6, 21, 79, 311, 1265, 5275, 22431,...	A033321

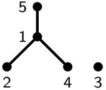
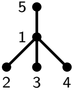
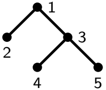
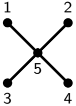
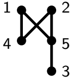
More POPs of length 4 with longest chain of size 3

POP	Sequence (beginning with $n = 1$)	OEIS
	1, 2, 6, 22, 90, 394, 1806, 8558, 41586,...	A006318
	1, 2, 6, 22, 90, 396, 1837, 8864, 44074,...	A053617
	1, 2, 6, 22, 90, 395, 1823, 8741, 43193,...	A165546

POPs of length 5

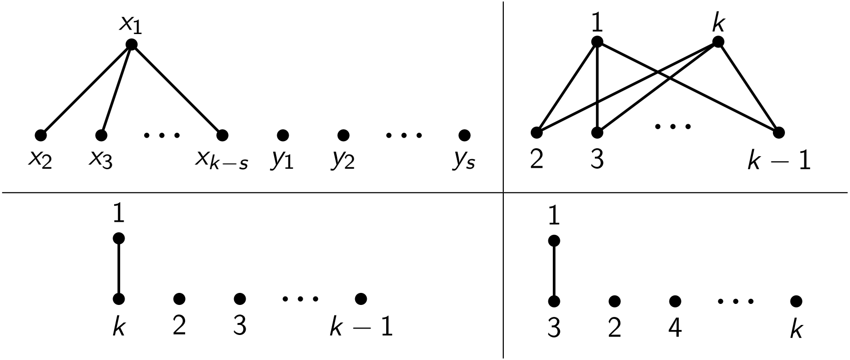
POP	Sequence (beginning with $n = 1$)	OEIS
	1, 2, 6, 24, 60, 150, 399, 1145,...	A276838
	1, 2, 6, 24, 60, 120, 210, 336,...	A007531
	1, 2, 6, 24, 96, 384, 1536, 6144,...	A084509
	1, 2, 6, 24, 108, 504, 2376, 11232,...	A094433
	1, 2, 6, 24, 100, 408, 1624, 6336,...	A094012
	1, 2, 6, 24, 115, 618, 3591, 22088,...	A128088

Conjectured OEIS connections for POPs of length 5

POP	Sequence (beginning with $n = 1$)	OEIS
	1, 2, 6, 24, 110, 540, 2772, 14704,...	A216879
	1, 2, 6, 24, 114, 600, 3372, 19824,...	A054872
	1, 2, 6, 24, 112, 568, 3032, 16768,...	A118376
	1, 2, 6, 24, 116, 632, 3720, 23072,...	A212198
	1, 2, 6, 24, 114, 598, 3336, 19402,...	A228907

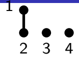
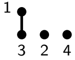
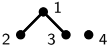
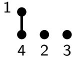
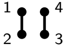
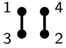
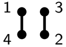
Our general results

Enumeration of POP-avoiding permutations is known for the following posets, where $k \geq 1$, $0 \leq s \leq k$, $\{x_1, x_2, \ldots, x_{k-s}\} = \{t, t+1, \ldots, t+k-s-1\}$ for some t, $1 \leq t \leq s+1$, and
$\{y_1, y_2, \ldots, y_k\} = \{1, 2, \ldots, t-1\} \cup \{t+k-s, t+k-s+1, \ldots, k\}$:

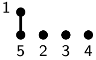
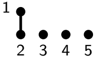
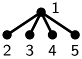
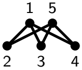
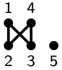


Also, applying the complement on labels of a POP p of size k (replacing a label x by $k+1-x$), or flipping p up-side-down, gives an equivalent poset.

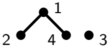
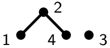
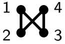
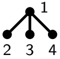
New PP enumeration results (generating functions)

	$\frac{1-2x+2x^2+2x^3-x^4}{(1-x)^3}$
	$\frac{1-x-x^2+3x^3+x^4}{(1-x-x^2)^2}$
	$\frac{1-3x+2x^2+2x^3}{(1-2x)^2}$
	$\frac{1}{1-x-x^2-3x^3-x^4}$
	$\frac{1-4x+5x^2}{(1-x)(1-2x)^2}$
	$\frac{1-x-2x^2-2x^3}{1-2x-2x^2-2x^3}$
	$\frac{(1-x)^3}{1-4x+5x^2-4x^3}$

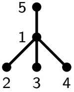
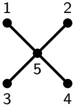
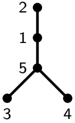
New PP enumeration results (generating functions)

	$\frac{1-3x+x^2}{1-4x+3x^2-x^3}$
	$\frac{1-x^2}{1-x-2x^2-2x^3-12x^4-8x^5+2x^6+5x^7+x^8}$
	$\frac{1-3x+4x^2+9x^4-7x^5+2x^6}{(1-x)^4}$
	$\frac{1-3x-2x^2-2x^3}{1-4x}$
	$\frac{1-5x+2x^2}{1-6x+6x^2}$
	$\frac{1-7x+14x^2-6x^3+4x^4}{(1-4x+2x^2)^2}$

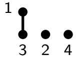
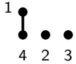
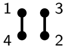
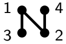
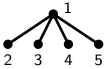
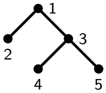
Open bijective problems on POPs and other patterns

POP	OEIS	Equinumerous structures
	A111281	permutations avoiding the patterns 2413, 2431, 4213, 3412, 3421, 4231, 4321, 4312
	A111282	permutations avoiding the patterns 1432, 2431, 3412, 3421, 4132, 4231, 4312, 4321
	A111277	permutations avoiding the patterns 2413, 4213, 2431,4231, 4321; also, permutations avoiding the patterns 3142, 3412, 3421, 4312, 4321
	A006012	permutations avoiding the vincular patterns $1\underline{32}4$, $1\underline{42}3$, $2\underline{31}4$, $2\underline{41}3$; see [Y. Biers-Ariel. The number of permutations avoiding a set of generalized permutation patterns, *J. Integer Sequences* **20** (2017), Article 17.8.3.]
	A025192	permutations $\pi_1 \cdots \pi_{3n}$ avoiding the patterns 231, 312, 321 and satisfying $\pi_{3i+1} < \pi_{3i+2}$ and $\pi_{3i+1} < \pi_{3i+3}$ for all $0 \leq i < n$. Equivalently, 2-ary shrub forests of n heaps avoiding the patterns 231, 312, 321; see [D. Bevan, D. Levin, P. Nugent, J. Pantone, L. Pudwell, M. Riehl, M. Tlachac. Pattern avoidance in forests of binary shrubs. *Discr. Math. Theor. Comp. Sci.* **18:2** (2016), #8.]

More bijective problems on POPs and other patterns

POP	OEIS	Equinumerous structures
	A054872	permutations avoiding the patterns 12345, 13245, 21345, 23145, 31245, 32145; note that avoiding these patterns is the same as avoiding the POP $\{5>4, 4>1, 4>2, 4>3\}$
	A212198	permutations avoiding the marked mesh pattern M(2,0,2,0); see [S. Kitaev, J. Remmel. Quadrant marked mesh patterns. *J. Integer Seq.* **15(4)** (2012), Art. 12.4.7, 29.]; these permutations are proved to be in bijection with pattern-avoiding involutions $\mathbf{I}_n(>,\neq,>)$; see [M. Martinez, C. Savage. Patterns in Inversion Sequences II: Inversion Sequences Avoiding Triples of Relations. *J. Integer Seq.* **21** (2018), Article 18.2.2.]
	A224295	permutations avoiding the patterns 12345 and 12354; note that avoiding these patterns is the same as avoiding the POP $\{1>2, 2>3, 3>4, 3>5\}$

Other open bijective problems

POP	OEIS	Equinumerous structures
	A045925	levels in all compositions of $n+1$ with only 1's and 2's
	A214663	n-permutations for which the partial sums of signed displacements do not exceed 2
	A232164	Weyl group elements, not containing ...
	A271897	sum of all second elements at level n of the TRIP-Stern sequence corresponding to the permutation triple (e, e, e)
	A052544	compositions of $3n+1$ into parts of the form $3m+1$
	A084509	number of ground-state 3-ball juggling sequences of period n
	A118376	series-reduced enriched plane trees of weight n; also, trees of weight n, where nodes have positive integer weights and the sum of the weights of the children of a node is equal to the weight of the node