On k-11-representable graphs

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June 30th, 2020

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A **motivation** to study various encodings of graphs by words is the hope, for a given (difficult) problem on graphs, to be able to find a **suitable encoding** that would allow to translate the problem on graphs to an **easier** problem on words, and solve it. Such an encoding does **not** have to be **optimal in size**.

**Example: Prüfer codes (sequences) to encode labelled trees (1918)**

Provides a proof of **Cayley’s formula** \( n^{n-2} \) to enumerate labelled trees on \( n \) vertices. Remove the leaf with the **smallest label** and record its neighbour: 4445 (the last neighbour does not need to be recorded).
All graphs considered by us are simple (no loops, no multiple edges).

A graph $G = (V, E)$ is word-representable if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$, $x \neq y$, alternate in $w$ if and only if $xy \in E$. ($w$ must contain each letter in $V$).

$w$ is a word-representant. We say that $w$ represents $G$.

We deal with unlabelled graphs. However, to apply the definition, we need to label graphs. Any labelling of a graph is equivalent to any other labelling because letters in $w$ can always be renamed.
Word-representable graphs

Example

The graph \begin{tikzpicture}

\draw (-0.5,0) node[anchor=north] {1} -- (0,0) node[anchor=north] {2} -- (0.5,0) node[anchor=north] {4} -- (0,1) node[anchor=south] {3} -- (-0.5,0);
\end{tikzpicture}

is word-representable.

Indeed, 2 4 can be represented by 1213423.

Remark

The class of word-representable graphs is hereditary. That is, removing a vertex \( v \) in a word-representable graph \( G \) results in a word-representable graph \( G' \).
The best ways to learn about the subject

Word-representable graph
From Wikipedia, the free encyclopedia

In the mathematical field of graph theory, a word-representable graph is a graph that can be characterized by a word (or se...
Several facts about word-representable graphs

- Introduced by the presenter in 2004 based on the joint research with Steven Seif on the celebrated Perkin’s semigroup
- Includes several well-known graph classes, e.g. 3-colorable graphs, circle graphs and comparability graphs
- Not all graphs are word-representable. The minimum non-word-representable graph is the wheel graph $W_5$ on 6 vertices
- There is a useful characterization of word-representable graphs in terms of certain graph orientations called semi-transitive orientations
- Recognizing word-representability is an NP-complete problem
- Enumeration is known for $\leq 11$ vertices (3 years of computations!)
- Word-representants can be assumed to have the same number of occurrences of each letter and be of length at most $2n^2$ (for a graph on $n$ vertices)
No edge between $x$ and $y$  $=$  an occurrence of either $xx$ or $yy$
$=$  an occurrence of the pattern $11$

- An idea of Jeff Remmel (2017): Why not to allow a few occurrences of the pattern $11$ when defining an edge?
- A graph $G = (V, E)$ is $k$-11-representable if there exists a word $w$ such that $xy \in E$ iff $w$ restricted to $x$ and $y$ contains at most $k$ occurrences of the pattern $11$.
- $k$ in the definition of $k$-11-representable graphs can be thought of as the degree of tolerance
- Note that $0$-11-representation $=$ word-representability
- $k$-u-representation of graphs is defined similarly to the case of $u = 11$ by letting $u$ be any binary pattern
Facts about k-11-representable graphs

The following facts were established in “G.-S. Cheon, J. Kim, M. Kim, S. Kitaev, A. Pyatkin. On k-11-representable graphs. J. of Combin. (2019)”

- **k-11-representability** implies \((k + 1)\)-11-representability
- The class of **0-11-representable graphs** is **strictly** inside of the class of **1-11-representable graphs** (in particular, **all** non-word-representable graphs on up to 7 vertices are 1-11-representable)
- For any graph \(G\), there exists a \(k\) such that \(G\) is **k-11-representable**
- In fact, **all** graphs are **2-11-representable graphs**
- A graph is an **interval graph iff** it is 1-11-representable by a word containing two copies of each letter; such words characterise **circle graphs** in the 0-11-representation case
The hierarchy related to k-11-representable graphs

Open problem: Can we 1-11-represent every graph? In other words, is \( k = 1 \) equal to \( k = 2 \)?

Hope: The answer to the question is “Yes!” because 2-11-representation of graphs is achieved by a concatenation of permutations (giving just transitive graphs in the 0-11-representation case!); however, we can mix letters in words in a more sophisticated way...