Enumerating Symmetric and Asymmetric Peaks in Dyck paths

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Symmetric Peaks in Words (W. Asawly, 2018)

A **word** ω of length n over an alphabet $[k] = \{1, \ldots, k\}$ contains a **peak**, if there is $i \in \{2, \ldots, n-1\}$ such that $\omega_{i-1} < \omega_i$, and $\omega_{i+1} < \omega_i$.

Example

$w = 2131125243$

Peaks: $w = 21\underline{\mathbf{3}}112\underline{\mathbf{5}}2\underline{\mathbf{4}}3$

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$\omega_{i-1} = \omega_{i+1} < \omega_i$, and
we say that $\omega_{i-1}, \omega_i, \omega_{i+1}$ is **asymmetric peak**, if
$\omega_{i-1} \neq \omega_{i+1}$ and both $< \omega_i$.

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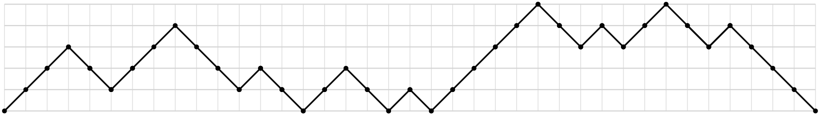
What about symmetric peaks in Dyck path?

- A **Dyck word** is a word in the letters X and Y with as many X's as Y's and in which no initial segment has more Y's than X's.
- A **peak** is a subword of the form XY

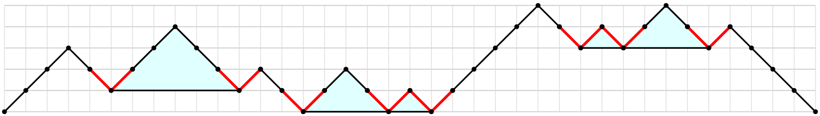
- **Dyck path** associated to the Dyck word



$$XXXYYXXXYYYXYYXXYYXYXXXXXYYXYXXYYXYYYY$$

What about symmetric peaks in Dyck path?

- A peak is **symmetric** if the maximal pyramid having the peak is not preceded by X and not follows a Y.
  (That is, a peak is *symmetric* if the valleys determining the maximal pyramid containing the peak are the at same level.)



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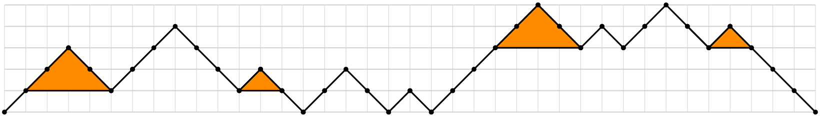
What about symmetric peaks in Dyck path?

- A peak is **symmetric** if the maximal pyramid having the peak is not preceded by X and not followed by X.

- A peak is **asymmetric** if the maximal pyramid having the peak is either preceded by X or follows a Y.



$$XXXYYXXXYYYXYYXXYYXYXXXXXYYXYXXYYXYYYY$$

Number of symmetric peaks in Dyck paths

- $\mathfrak{D}_n$ is the set of Dyck paths of length $2n$.
- $g_n := \#$ of symmetric peaks in $\mathfrak{D}_n$.

Theorem

The recurrence relations of g_n is given by

$$g_n = C_{n-1} + g_{n-1} + \sum_{k=1}^{n-1} \left(\left(g_k - 2 \sum_{i=1}^{k-1} C_i \right) C_{n-k-1} + C_k \, g_{n-k-1} \right),$$

with initial values $g_0 = 0$ and $g_1 = 1$, where C_n is the n-th Catalan number.

Example. Number of symmetric peaks in Dyck paths

$$\{g_n\}_{n\geq 1} = \{1,\ 3,\ 8,\ \mathbf{23},\ 72,\ 240,\ 834,\ 2979,\ 10844,\ 40016,\ldots\}$$

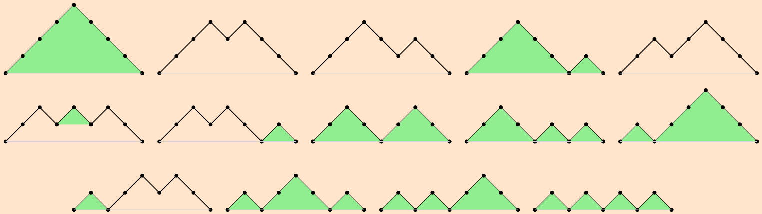


Figure: Dyck paths of length 2(4) and their symmetric peaks.

Number of symmetric peaks in Dyck paths

Theorem

The **generating function** *of the number of symmetric peaks in the family of Dyck paths is given by*

$$G(x) = \frac{1 - 5x + (x-1)\sqrt{1-4x}}{2(x-1)\sqrt{1-4x}}.$$

$$g_n = 2\sum_{k=0}^{n} \binom{2k}{k} - \frac{5}{2}\binom{2n}{n}.$$

Theorem

The sequence g_n *has the* **asymptotic approximation**

$$g_n \sim \frac{2^{2n}}{3\sqrt{\pi n}} \left(\frac{1}{2} + \frac{61}{48n} + \frac{899}{768n^2} + \frac{38125}{18432n^3} + O(n^{-4}) \right).$$

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Proportion of symmetric peaks in Dyck path

The total number of paths in $\mathfrak{D}_n$ with exactly k peaks is given by the Narayana number $N(n,k)$, $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. Therefore, the total number of peaks in $\mathfrak{D}_n$ is

$$t_n := \sum_{k=0}^{n} kN(n,k) = \binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}, \quad n \geq 1.$$

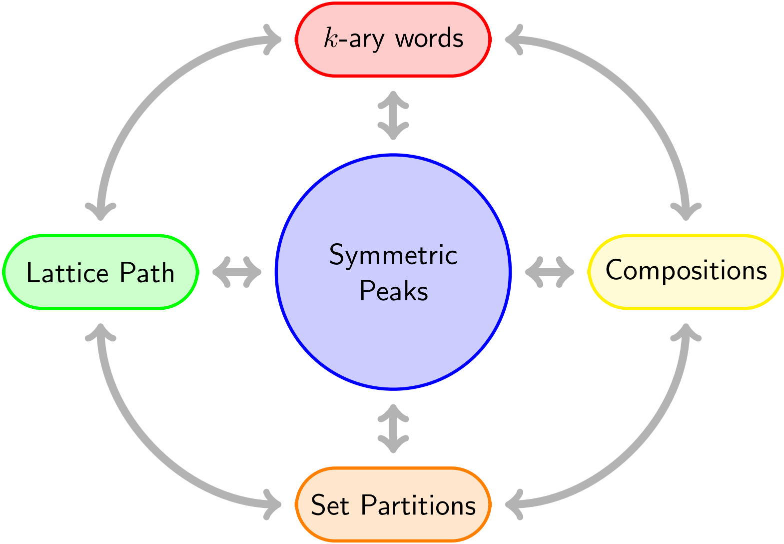
Theorem

The proportion between the total number of symmetric peaks and peaks has the asymptotic approximation

$$\frac{g_n}{t_n} \sim \frac{16}{3}\left(\frac{1}{16} + \frac{1}{6n} + \frac{1}{6n^2} + \frac{5}{18n^3} + O(n^{-4})\right).$$

This proportion is approximately 0.33418.

Future Works



Thanks!