2-AVOIDANCE

Murray Elder, UTS
Permutation Patterns in cyberspace 2020
Interactive video at
https://uts.h5p.com/content/1291033806121141429
Definition (2-containment $^{1,2}$)

Let $\sigma$ be a permutation and $F, G \subseteq S^\infty$.

We say that $\sigma$ 2-contains $(F, G)$ if there exists $\gamma <_{\text{subperm}} \sigma$ such that
- $\text{red}(\gamma) \in F$ and
- there is no $\delta <_{\text{subperm}} \sigma$ such that $\gamma <_{\text{subperm}} \delta$ and $\text{red}(\delta) \in G$.

Informally we think of the set $G$ as patterns which can potentially save a permutation from being forbidden by $F$.

Eg: 15234 2-contains (\{123\}, \{1423\})

$^{1}\text{red}(\alpha)$ — permutation obtained by replacing the $i$th smallest entry of $\alpha$ by the integer $i$

$^{2}S^{\infty}$ — set of all reduced permutations
A permutation 2-avoids \((F, G)\) if it does not 2-contain \((F, G)\). By propositional logic:

**Definition (2-avoidance)**

Let \(\sigma\) be a permutation and \(F, G \subseteq S^\infty\).

\(\sigma\) 2-avoids \((F, G)\) if for all \(\gamma \prec_{\text{subperm}} \sigma\), if \(\text{red}(\gamma) \in F\) then there exists \(\delta \prec_{\text{subperm}} \sigma\) such that \(\gamma \prec_{\text{subperm}} \delta\) and \(\text{red}(\delta) \in G\): \(\delta\) saves \(\gamma\).

Eg: 15423 2-avoids \((\{123\}, \{1423\})\)

We denote the set of all permutations in \(S^\infty\) which 2-avoid \((F, G)\) by \(\text{Av}_2(F, G)\).
$F = \{3241\}, G = \{41352\}$. Does the perm 2-contain or 2-avoid $(F, G)$?

- 143562

- 152463
$F = \{1\}, G = \{12, 21\}$. Then $\text{Av}_2(F, G) = S^\infty \setminus \{1\}$.

This shows that the growth of (proper) 2-avoidance sets can be factorial (in contrast to$^3$)

Question: what growth rates are possible for 2-avoidance?

- super-exponential but sub-factorial?
- super-polynomial but sub-exponential? (in contrast to$^4$)

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Barred pattern avoidance:

A permutation avoids $4 \overline{1} 352$ means if it has any subsequence order isomorphic to 3241, that subsequence must be a part of a subsequence order isomorphic to 41352.

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5. Deterministic pop stack: sorting device with two operations - push: move a token from the input to the top of the stack - pop: move the entire stack contents to the output - always push unless the token on the top of the stack is smaller in value than the token to be pushed from the input.
Theorem (Pudwell and Smith\textsuperscript{6})

The set of 2-pass pop stack sortable permutations is equal to

\[ \text{Av}_B (\{2341, 3412, 3421, 4123, 4231, 4312, 3241, 41352, 41352\}) \]

If we were to characterise 3-pop stack sortable permutations as those avoiding some list containing $463\bar{1}572$ and $473\bar{1}562$, then we would be mistaken:

$473\bar{1}562$ does not avoid this list since it fails to avoid $463\bar{1}572$. 
2-PASS AGAIN

Theorem (Pudwell and Smith)

The set of 2-pass pop stack sortable permutations is equal to

$\text{Av}_B (\{2341, 3412, 3421, 4123, 4231, 4312, 3241, 41352, 413\bar{5}2\})$

becomes

Theorem (Pudwell and Smith)

The set of 2-pass pop stack sortable permutations is equal to

$\text{Av}_2(\{2341, 3412, 3421, 4123, 4231, 4312, 3241, 4132\}, \{41352\})$
**Theorem (E, Goh)**

For each $k \in \mathbb{N}$ there exist finite sets $F_k, G_k \subseteq S^\infty$ such that $\sigma$ is sortable by $k$-passes through a pop stack iff $\sigma$ 2-avoids $(F_k, G_k)$.

More details in Andrew’s talk tomorrow.

This\(^8\) answers Claesson and Guðmundsson’s\(^9\) question: is there a “useful permutation pattern characterization of the $k$-pop stack-sortable permutations”?

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\(^7\) Elder and Goh, “$k$-pop stack sortable permutations and 2-avoidance”, 2019.

\(^8\) we think, if you like 2-avoidance

\(^9\) Claesson and Guðmundsson, “Enumerating permutations sortable by $k$ passes through a pop-stack”, 2018.
THANKS