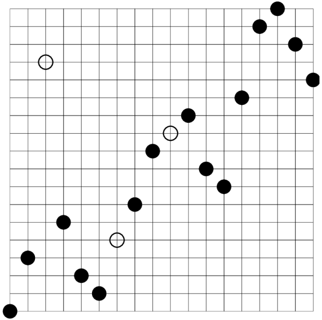
On permutation patterns with constrained gap sizes



Stoyan Dimitrov

University of Illinois at Chicago

2020 Permutation Patterns Virtual Workshop

Definition 1

$$\pi = \pi_1 \pi_2 \cdots \pi_n \in Av(1\Box 2),$$
if there is no $0 < i < j - 1 < n$ with $\pi_i < \pi_j$,
i.e., the $\Box$ denotes a gap of minimal size 1.

Example: 24 is an occurrence of $1\Box 2$ in $\pi = 25341$, but 34 is not.

A motivating question

Definition 1

$$\pi = \pi_1\pi_2\cdots\pi_n \in Av(1\Box 2),$$

if there is no $0 < i < j-1 < n$ with $\pi_i < \pi_j$,

i.e., the $\Box$ denotes a gap of minimal size 1.

Example: 24 is an occurrence of $1\Box 2$ in $\pi = 25341$, but 34 is not.

Theorem 1 (Simion and Schmidt, [23])

$$|Av_n(1\Box 2)| = F_{n+1},$$

i.e., the $(n+1)$*st Fibonacci number.*

Definition 1

$$\pi = \pi_1\pi_2\cdots\pi_n \in Av(1\square 2),$$

if there is no $0 < i < j - 1 < n$ with $\pi_i < \pi_j$,

i.e., the $\square$ denotes a gap of minimal size 1.

Example: 24 is an occurrence of $1\square 2$ in $\pi = 25341$, but 34 is not.

Theorem 1 (Simion and Schmidt, [23])

$$|Av_n(1\square 2)| = F_{n+1},$$

i.e., the $(n+1)$st Fibonacci number.

What about $Av_n(1\square_r 2)$, where $\square_r$ denotes a gap of minimal size r?

Definition 2 (Distant pattern (DP))

A Permutation Pattern (PP) containing $\square_r$ symbols, where $r \geq 0$.
($\square$ will denote $\square_1$).

Distant patterns (DPs)

Definition 2 (Distant pattern (DP))

A Permutation Pattern (PP) containing $\square_r$ symbols, where $r \geq 0$.
($\square$ will denote $\square_1$).

Examples:
- 235 is an occurrence of $1\square_2 23$ in $\pi = 7264135$, but 245 is not.
- 523 is an occurrence of $\square 31 \square_3 2$ in $\pi = 6521473$, but 623 is not.

Definition 2 (Distant pattern (DP))

A Permutation Pattern (PP) containing $\square_r$ symbols, where $r \geq 0$.
($\square$ will denote $\square_1$).

Examples:
- 235 is an occurrence of $1\square_2 23$ in $\pi = 7264135$, but 245 is not.
- 523 is an occurrence of $\square 31 \square_3 2$ in $\pi = 6521473$, but 623 is not.

In fact, every DP can be written as:

$$\square_{r_0} q_1 \square_{r_1} q_2 \square_{r_2} \cdots \square_{r_{k-1}} q_k \square_{r_k},$$

where $r_i \geq 0$ and $q_1 q_2 \cdots q_k$ is a k-permutation.

Other Definitions

- $Av_n(q)$ and $Av_n(\Pi)$ - the set of permutations in S_n avoiding the pattern q and all the patterns in the set Π, respectively.
- *gap* and *gap size* - the space between two consecutive letters of a pattern and its size.
- *size of a DP* - the number of non-$\square$ letters in it.

Other Definitions

- $Av_n(q)$ and $Av_n(\Pi)$ - the set of permutations in S_n avoiding the pattern q and all the patterns in the set Π, respectively.
- *gap* and *gap size* - the space between two consecutive letters of a pattern and its size.
- *size of a DP* - the number of non-$\square$ letters in it.
- *vincular DPs* - DPs with tight constraints, e.g., $\underline{1\square_4 2}3$ denotes the pattern 123 with gap size exactly 4 between the letters 1 and 2.
- *uniform DPs* - the same minimal gap size constraint for all the gaps. If $q = q_1 \dots q_k$, then denote $dist_r(q) = q_1\square_r \dots \square_r q_k$.

Some previous work on DPs

- 2005, "Partially ordered generalized patterns", S.Kitaev [15]
  - a generalization of DPs considered in a series of papers [10, 13, 14, 15, 16, 17, 18], including a recent one by Kitaev and Gao [9].
- 2007, "Distanced patterns", PhD Thesis, G.Firro [6]
  - another generalization of DPs which allows requiring a gap size to be $\leq r$.
  - based on works with Mansour [7, 8] establishing a bijection between the permutations in $Av_n(1\square 23)$ and certain set of dissections of an $(n+2)$-gon.
- 2016, Hopkins and Weiler ([11])
  - their work imply a result on uniform distant patterns.

Summary of the results

I. Two basic facts about DPs.

II. DPs of size 2. Answer to the motivating question.

III. DPs of size 3 and the enumeration of $|Av_n(1\square 3\square 2)|$.

I. Two basic facts about DPs.

II. DPs of size 2. Answer to the motivating question.

III. DPs of size 3 and the enumeration of $|Av_n(1\square 3\square 2)|$.

IV. Vincular DPs and a surprising relation between $Av_n(123)$ and $Av_n(132)$.

V. Proofs of two conjectures using DPs.

VI. Stanley-Wilf analogues for DPs and some open questions.

Reducing avoidance of DPs to avoidance of classical patterns:

Theorem 2

The avoidance of $q = \square_{r_0} q_1 \square_{r_1} q_2 \square_{r_2} \cdots \square_{r_{k-1}} q_k \square_{r_k}$, *where* $\sum_{j=0}^{k} r_j = S$, *is equivalent to the simultaneous avoidance of* $\frac{(S+k)!}{k!}$ *classical patterns of size* $S+k$.

What if a DP starts or ends with squares?:

Theorem 3

For any $r_1, r_2 > 0$ *and a distant pattern* q,

$$|Av_n(\Box_{r_1} q \Box_{r_2})| = \frac{n!}{(n-r)!} |Av_{n-r}(q)|,$$

where $r := r_1 + r_2$.

Obviously, $|Av_n(2\Box_r 1)| = |Av_n(1\Box_r 2)|$ for each $r \geq 0$. Furthermore,

Theorem 4 (see A276837 in OEIS)

The permutations in $Av_n(2\Box_r 1)$ are in one-to-one correspondence with the permutations in S_n for which, when written in a cycle notation, any two elements in a cycle differ by at most r.

Obviously, $|Av_n(2\Box_r 1)| = |Av_n(1\Box_r 2)|$ for each $r \geq 0$. Furthermore,

Theorem 4 (see A276837 in OEIS)

The permutations in $Av_n(2\Box_r 1)$ are in one-to-one correspondence with the permutations in S_n for which, when written in a cycle notation, any two elements in a cycle differ by at most r.

Proof: We construct a bijection.

Example:
$n = 9$, $r = 3$,
$352149867 \in Av_9(2\Box_3 1)$ maps to $(134)(25)(6798)$,
which does not have a, b in the same cycle, s.t. $|a - b| > 3$.

Obviously, $|Av_n(2\Box_r 1)| = |Av_n(1\Box_r 2)|$ for each $r \geq 0$. Furthermore,

Theorem 4 (see A276837 in OEIS)

The permutations in $Av_n(2\Box_r 1)$ are in one-to-one correspondence with the permutations in S_n for which, when written in a cycle notation, any two elements in a cycle differ by at most r.

Proof: We construct a bijection.

Example:
$n = 9, r = 3,$
$352149867 \in Av_9(2\Box_3 1)$ maps to $(134)(25)(6798)$,
which does not have a, b in the same cycle, s.t. $|a - b| > 3$.

This answers our initial motivating question!

Is there a relation between the sets $Av_{n-1}(12)$ and $Av_n(1\square 2)$?

Is there a relation between the sets $Av_{n-1}(12)$ and $Av_n(1\Box 2)$?

Try to see such using a simple algorithm:

1. Take π containing 12 and $j \in [n]$.
2. Insert j immediately after the 1 in the leftmost 12-pattern occurrence in π.
3. Increment all the numbers $\geq j$.

We get an n-permutation containing $1\Box 2$.

Is there a relation between the sets $Av_{n-1}(12)$ and $Av_n(1\square 2)$?

Try to see such using a simple algorithm:

1. Take π containing 12 and $j \in [n]$.
2. Insert j immediately after the 1 in the leftmost 12-pattern occurrence in π.
3. Increment all the numbers $\geq j$.

We get an n-permutation containing $1\square 2$.

Let $C_n(q)$ be the set of permutations containing q.
The algorithm above defines a map

$$g : C_{n-1}(12) \times [n] \to C_n(1\square 2).$$

Example:
$\pi = 3412 \in C_4(12), j = 2,$
$g(3412, 2) = 42513 \in C_5(1\Box 2).$

Example:
$\pi = 3412 \in C_4(12), j = 2,$
$g(3412, 2) = 42513 \in C_5(1\Box 2).$

One problem: the map g is neither injective, nor surjective.

Example:
$\pi = 3412 \in C_4(12), j = 2,$
$g(3412, 2) = 42513 \in C_5(1\square 2).$

One problem: the map g is neither injective, nor surjective.

However, if $t(\omega) = |g^{-1}(\omega)|$ denotes the number of different tuples $(\pi, j) \in C_{n-1}(12) \times [n]$, such that $g(\pi, j) = \omega$,
then the following facts hold:

i. $t(\omega) \leq 2$, for each $\omega \in C_n(1\square 2)$.

ii. $t(\omega) = 2$ for exactly:

$$\sum_{j=3}^{n-1} (j-2)(n-j)(n-j)!$$

permutations $\omega \in C_n(1\square 2)$.

iii. $t(\omega) = 0$ for exactly:

$$\sum_{k=3}^{n-2} (F_{n-k+1} - 1)k(k-2)(k-2)!$$

permutations $\omega \in C_n(1\square 2)$.

Using the map g, the facts $(i)-(iii)$ and Inclusion-Exclusion, we get:

$$|Av_n(1\square 2)| = F_{n+1} = n + \sum_{k=1}^{n-3}\big(n-(k+2)F_{n-(k+1)}\big)\cdot k\cdot k!.$$

Consider DPs of size 3 with one square.

According to works of Firro and Mansour [6, 7, 8], there is a single Wilf-equivalent class:

$$|Av_n(xy\Box z)| = \sum_{k\geq 0} \frac{1}{n-k}\binom{2n-2k}{n-1-2k}\binom{n-k}{k},$$

for every $xyz \in S_3$.

Consider DPs of size 3 with two squares of the kind $x\Box y\Box z$.

There are two Wilf-equivalent classes for these patterns:

Class 1: $\{1\Box 2\Box 3, 3\Box 2\Box 1\}$

and

Class 2: $\{1\Box 3\Box 2, 2\Box 3\Box 1, 2\Box 1\Box 3, 3\Box 1\Box 2\}$.

Consider DPs of size 3 with two squares of the kind $x\square y\square z$.

There are two Wilf-equivalent classes for these patterns:

Class 1: $\{1\square 2\square 3, 3\square 2\square 1\}$

and

Class 2: $\{1\square 3\square 2, 2\square 3\square 1, 2\square 1\square 3, 3\square 1\square 2\}$.

We will describe a way to find the ordinary generating function for $|Av_n(1\square 3\square 2)|$, i.e., how to enumerate the avoidance sequence for a pattern in Class 2.

III. DPs of size 3. $Av_n(1\square 3\square 2)$.

Let $q = 1\square 3\square 2$. Steps to find $G(x) = \sum_{n\geq 0} |Av_n(q)|x^n$:

1. Determine five possible block decompositions for $\pi = \alpha n\beta \in Av_n(q)$. [21, Mansour and Vainshtain]
2. Transform the decompositions from step 1 to the equation:

$$G = 1 + G(xH_1 + x^3H_1 + xH_2) + G^2(x - 2x^2 - x^3 - x^4), \quad (*)$$

where $H_i(x)$ are the generating functions for the number of permutations of size n in $\mathbb{H}_i$ ($i = 1, 2$) and where

$$\mathbb{H}_1 := \{\pi \mid \pi \in Av(q), |\pi| \geq 1 \text{ and } \pi \text{ does not have an occurrence of } 1\square\underline{32} \text{ ending at the last position of } \pi\}$$

and

$$\mathbb{H}_2 := \{\pi \mid \pi \in Av(q), |\pi| \geq 1 \text{ and } \pi \text{ does not have an occurrence of } \underline{13}\square 2 \text{ beginning at the first position of } \pi\}.$$

3. Define two additional gen.functions T_0 and T_1 and obtain the following relations, where $X \to \{A, B, C\}$ means that X can be written as an expression of A, B and C:
   - $H_2 \to \{G, H_1, T_0, T_1\}$
   - $H_1 \to \{G, H_2\}$
   - $T_0 \to \{G, H_2\}$
   - $T_1 \to \{G, H_1, H_2\}$

Note 1: Substituting the expressions for H_1, T_0, T_1 into the first listed expression, as well as the one for H_1 in (*) gives us a system of two relations between H_2 and G.

Note 2: A main tool for obtaining the above expressions is a lemma involving the so-called *push* operation:

$$push(\tau, \sigma) := \sigma[\tau, 1, 1, \cdots, 1].$$

i.e., considering the graph of σ, we replace its first element with the graph of τ (we push τ).

Note 3: The lemma shows that one can find the generating function for the number of permutations in the image of the map $push(\tau, \sigma)$ when $\tau \in U_1 \subseteq \mathbb{H}_2$, $\sigma \in U_2 \subseteq \mathbb{H}_2$ and under certain conditions for U_1, U_2.

Note 4: To obtain that $X \to \{A, B, C\}$, we usually decompose the set of permutations corresponding to X into blocks, where some blocks are of the kind $push(\tau, \sigma)$ for $\tau \in U_1 \subseteq \mathbb{H}_2$, $\sigma \in U_2 \subseteq \mathbb{H}_2$ and given U_1, U_2. We can find the generating functions for all blocks.

4. Following step 3, we obtain an equation $P(x, G(x)) = 0$, where P is a polynomial of $G(x)$ with coefficients - polynomials of x. P has 170 terms with the term of highest total degree being $x^{27}G^{12}$:

$$
\begin{aligned}
(x^2 + x^4) + G(x^6 + x^8) + G^2(-x^2 + 3x^3 - \cdots - x^{10}) + \\
\cdots + \ldots \\
+ G^{12}(-x^{12} - x^{13} - \cdots - 3x^{26} - x^{27}) = 0
\end{aligned}
$$

5. In order to get a closed form expression for $G(x)$, one could use a generalization of the Lagrange inversion formula [2, Baderier and Drmota].

IV. Vincular DPs.

We classify the patterns of the form $\underline{ab}\Box c$ and $a\Box\underline{bc}$:

Class 1	Class 2	Class 3
$\underline{12}\Box 3$	$1\Box\underline{32}$	$\underline{13}\Box 2$
$\underline{32}\Box 1$	$\underline{21}\Box 3$	$\underline{31}\Box 2$
$1\Box\underline{23}$	$\underline{23}\Box 1$	$2\Box\underline{31}$
$3\Box\underline{21}$	$3\Box\underline{12}$	$2\Box\underline{13}$

The Wilf-equivalent classes of $\underline{ab}\Box c$ and $a\Box\underline{bc}$ patterns.

IV. Vincular DPs.

We classify the patterns of the form $\underline{ab}\square c$ and $a\square\underline{bc}$:

Class 1	Class 2	Class 3
$\underline{12}\square 3$	$1\square\underline{32}$	$\underline{13}\square 2$
$\underline{32}\square 1$	$\underline{21}\square 3$	$\underline{31}\square 2$
$1\square\underline{23}$	$\underline{23}\square 1$	$2\square\underline{31}$
$3\square\underline{21}$	$3\square\underline{12}$	$2\square\underline{13}$

The Wilf-equivalent classes of $\underline{ab}\square c$ and $a\square\underline{bc}$ patterns.

Motivation: Every occurrence of the classical pattern 123 is an occurrence of a pattern in $\{\underline{123}, 1\square\underline{23}, \underline{12}\square 3, 1\square 2\square 3\}$.

Are these patterns avoided by less or by more permutations compared to their 132 counterparts?

Class 1:

Theorem 5
If $a_n = |Av_n(\underline{12}\square 3)|$, *then* $a_n = n!$ *for* $0 \leq n \leq 3$, *and for* $n \geq 4$,

$$a_n = a_{n-1} + (n-1)a_{n-2} + \frac{(n+1)(n-2)}{2}a_{n-3} + \sum_{i=4}^{n-1}\left(\binom{n}{i-1} - 1\right)a_{n-i} + (n-1).$$

Class 2:

Theorem 6

If $b_n = |Av_n(1\square\underline{32})|$*, then* $b_n = n!$ *for* $0 \le n \le 3$ *and for* $n \ge 4$*,*

$$b_n = b_{n-1} + (n-1)b_{n-2} + \frac{(n+1)(n-2)}{2}b_{n-3} + \sum_{i=2}^{n-3}\left(i\binom{n-2}{i} + \binom{n-1}{i-1}\right)b_{i-1} + (n-1).$$

Theorem 5 and Theorem 6 were proved via a method similar to Insertion-Encoding [1, Albert et al.].

Theorem 5 and Theorem 6 were proved via a method similar to Insertion-Encoding [1, Albert et al.].

Class 3:

Theorem 7

For all $n \in \mathbb{Z}^+$, $Av_n(\underline{13}\square 2) = Av_n(13\square 2)$, *which implies that* $|Av_n(\underline{13}\square 2)| = |Av_n(13\square 2)|$ *[A04912 in OEIS].*

Furthermore, somewhat complicated inductive arguments give us:

Theorem 8

For all $n \geq 5$, $|Av_n(1\square\underline{23})| < |Av_n(1\square\underline{32})|$.

Theorem 9

For all $n \geq 5$, $|Av_n(\underline{12}\square 3)| > |Av_n(\underline{13}\square 2)|$.

Furthermore, somewhat complicated inductive arguments give us:

Theorem 8

For all $n \geq 5$, $|Av_n(1\square\underline{23})| < |Av_n(1\square\underline{32})|$.

Theorem 9

For all $n \geq 5$, $|Av_n(\underline{12}\square 3)| > |Av_n(\underline{13}\square 2)|$.

We also have:

Theorem 10 (Elizalde, 2003, [5])

For all $n \geq 4$, $|Av_n(\underline{123})| > |Av_n(\underline{132})|$.

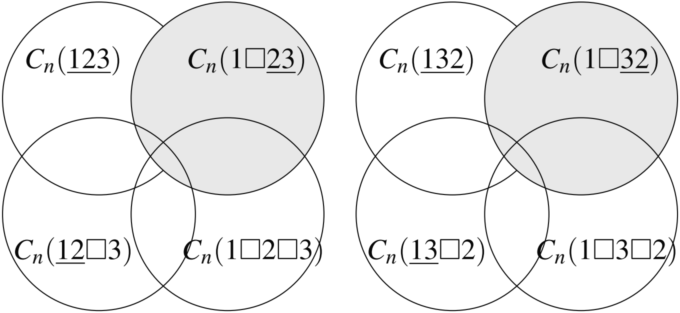
Theorem 11 (Hopkins and Weiler, 2016, [11])

For $n > 5$,

$$|Av_n(1\square 2\square 3)| > |Av_n(1\square 3\square 2)|.$$

These facts show that $1\square\underline{23}$ is the only pattern in $\{\underline{123}, 1\square\underline{23}, \underline{12}\square 3, 1\square 2\square 3\}$ avoided by fewer permutations of size n, compared to its counterpart $1\square\underline{32}$.

This is somewhat surprising given that $|Av_n(123)| = |Av_n(132)|$.



Venn diagrams for the n-permutations containing 123 and 132

DPs turned out to be useful when interpreting other results. Below is a conjecture of Kuszmaul that we proved using an interpretation with DPs:

Theorem 12 (conjectured by Kuszmaul, 2018, [20])

The generating function of

$$|Av_n(2431, 2143, 3142, 4132, 1432, 1342, 1324, 1423, 1243)|$$

is given by $C + x^3C$*, where* C *is the generating function for the Catalan numbers.*

Sketch of the proof:

Let $\Pi = \{2431, 2143, 3142, 4132, 1432, 1342, 1324, 1423, 1243\}$.

– Note that
$$\Pi = \{\Box 132, 132\Box, 1342\}.$$

– If $\sigma \in Av_n(132)$, then $\sigma \in Av_n(\Pi)$.

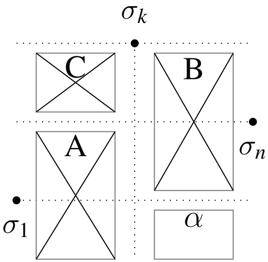
Sketch of the proof:

Let $\Pi = \{2431, 2143, 3142, 4132, 1432, 1342, 1324, 1423, 1243\}$.

- Note that
$$\Pi = \{\Box 132, 132\Box, 1342\}.$$
- If $\sigma \in Av_n(132)$, then $\sigma \in Av_n(\Pi)$.
- $|Av_n(132)|$ has generating function C [19, Knuth].
  It remains to find the generating function for those $\sigma \in Av_n(\Pi)$ containing 132.
- Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be one such permutation. Note that any occurrence of 132 in σ must be $\sigma_1\sigma_j\sigma_n$ for some $2 \le j \le n-1$.

Sketch of the proof (cont.):

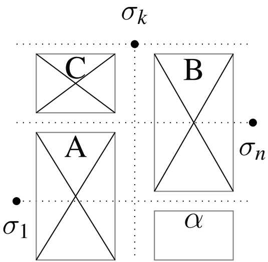
– Consider the graph of one such σ:



– We show that the regions A, B and C are empty.

Sketch of the proof (cont.):

– Consider the graph of one such σ:



– We show that the regions A, B and C are empty.

– Therefore, $\sigma = \sigma_1\sigma_2\alpha\sigma_n$, for some sequence of numbers α, where $\alpha < \sigma_1 < \sigma_n < \sigma_2$ and $\alpha \in Av(132)$.
This is where the x^3C term comes from!

Note: The same structure, for the decomposition of the permutations in $Av(\Pi)$, was recently found with a computer by Bean et al. [3, 2019] who used a particular algorithm called the *Struct algorithm*. .

Note: The same structure, for the decomposition of the permutations in $Av(\Pi)$, was recently found with a computer by Bean et al. [3, 2019] who used a particular algorithm called the *Struct algorithm*.

The second conjecture of Kuszmaul that we proved is given below:

Theorem 13

The generating function of

$$|Av_n(2431, 2413, 3142, 4132, 1432, 1342, 1324, 1423)|$$

is given by $C(1+x^3C)$*, where* C *is the generating function for the Catalan numbers.*

.

Note: The same structure, for the decomposition of the permutations in $Av(\Pi)$, was recently found with a computer by Bean et al. [3, 2019] who used a particular algorithm called the *Struct algorithm*.

The second conjecture of Kuszmaul that we proved is given below:

Theorem 13

The generating function of

$$|Av_n(2431, 2413, 3142, 4132, 1432, 1342, 1324, 1423)|$$

is given by $C(1+x^3C)$*, where* C *is the generating function for the Catalan numbers.*

We have found interpretations with DPs to four of the other conjectures stated in the same work [20].

Since avoidance of DPs is equivalent to avoidance of multiple classical patterns (see Theorem 2), it is easy to prove the following Stanley-Wilf type result for DPs:

Theorem 14

For any distant pattern q, there exists a constant $c_q > 0$, such that

$$\sqrt[n]{|Av_n(q)|} \underset{n\to\infty}{\longrightarrow} c_q.$$

Since avoidance of DPs is equivalent to avoidance of multiple classical patterns (see Theorem 2), it is easy to prove the following Stanley-Wilf type result for DPs:

Theorem 14

For any distant pattern q, there exists a constant $c_q > 0$, such that

$$\sqrt[n]{|Av_n(q)|} \underset{n\to\infty}{\longrightarrow} c_q.$$

Recall that $dist_r(q) = q_1\Box_r \ldots \Box_r q_k$ for a classical pattern $q = q_1 \ldots q_k$.

What about $\lim \sqrt[n]{|Av_n(dist_r(q))|}$**, when** r **increases with** n**?**

First, let us consider the case when the gap size constraint r is a fixed fraction of n:

Proposition 1

For any given classical pattern q, there exist constants $c > 0$ *and* $0 < c_1 < 1$*, such that*

$$\sqrt[n]{|Av_n(dist_r(q))|} \xrightarrow[r=\lfloor c_1 n \rfloor]{n\to\infty} c$$

First, let us consider the case when the gap size constraint r is a fixed fraction of n:

Proposition 1

For any given classical pattern q, there exist constants $c > 0$ *and* $0 < c_1 < 1$, *such that*

$$\sqrt[n]{|Av_n(dist_r(q))|} \xrightarrow[r=\lfloor c_1 n \rfloor]{n\to\infty} c$$

We refute this Proposition using Theorem 4.
In particular, we show that $Av_n(1\Box_r 2) = \Omega(C^n)$ for any $C > 0$.

What if r is $o(n)$, e.g., a function of the kind n^{c_2}, for $0 < c_2 < 1$?

Conjecture 1

For any given classical pattern q, there exist $c_q > 0$ *and* $0 < c_2 < 1$*, such that*

$$\sqrt[n]{|Av_n(dist_r(q))|} \underset{r=\lfloor n^{c_2}\rfloor}{\overset{n\to\infty}{\longrightarrow}} c_q$$

What if r is $o(n)$, e.g., a function of the kind n^{c_2}, for $0 < c_2 < 1$?

Conjecture 1

For any given classical pattern q, there exist $c_q > 0$ and $0 < c_2 < 1$, such that

$$\sqrt[n]{|Av_n(dist_r(q))|} \xrightarrow[r=\lfloor n^{c_2} \rfloor]{n\to\infty} c_q$$

A few other natural conjectures on DPs are listed in the paper: https://arxiv.org/abs/2002.12322

Thanks for the attention!

References I

- Albert, M. H., Linton, S., Ruŝkuc, N. (2005). The insertion encoding of permutations. Electronic Journal of Combinatorics, 12(1), 47.
- Banderier, C., Drmota, M. (2015). Formulae and asymptotics for coefficients of algebraic functions. Combinatorics, Probability and Computing, 24(1), 1-53.
- Bean, C., Gudmundsson, B., Ulfarsson, H. (2019). Automatic discovery of structural rules of permutation classes. Mathematics of Computation, 88(318), 1967-1990.
- Bean, C. (2019). Automatic methods for enumerating permutation classes. Lecture Notes. Pre-workshop "Permutation Patterns" 2019.

References II

- Elizalde, S., Noy, M. (2003). Consecutive patterns in permutations. Advances in Applied Mathematics, 30(1-2), 110-125.
- Firro, G. (2007), Distanced Patterns. PhD Thesis. University of Haifa.
- Firro, G., Mansour, T. (2005). Restricted permutations and polygons. In The Third International Conference on Permutation Patterns (pp. 7-11).
- Firro, G., Mansour, T. (2006). Three-letter-pattern avoiding permutations and functional equations. Electronic Journal of Combinatorics, 13(1), 51.
- Gao, A. L., Kitaev, S. (2019). On partially ordered patterns of length 4 and 5 in permutations. arXiv preprint arXiv:1903.08946.

References III

- Heubach, S., Kitaev, S., Mansour, T. (2006). Partially ordered patterns and compositions. arXiv preprint math/0610030.
- Hopkins, S., Weiler, M. (2016). Pattern avoidance in poset permutations. Order, 33(2), 299-310.
- Hou, Q. H., Mansour, T. (2006). Horse paths, restricted 132-avoiding permutations, continued fractions, and Chebyshev polynomials. Discrete applied mathematics, 154(8), 1183-1197.
- Kitaev, S., Mansour, T. (2003). Partially ordered generalized patterns and k-ary words. Annals of Combinatorics, 7(2), 191-200.
- Kitaev, S. (2005). Segmental partially ordered generalized patterns. Theoretical Computer Science, 349(3), 420-428.

References IV

- Kitaev, S. (2005). Partially ordered generalized patterns. Discrete Mathematics, 298(1-3), 212-229.
- Kitaev, S. (2007). Introduction to partially ordered patterns. Discrete Applied Mathematics, 155(8), 929-944.
- Kitaev, S. (2010). A survey on partially ordered patterns. Permutation Patterns, 376, 115-135.
- Kitaev, S. (2011). Patterns in permutations and words. Springer Science and Business Media.
- Knuth, D. (1973). The art of computer programming, volume 1, Fundamental Algorithms, Addison-Wesley, Reading, Massachusetts.

References V

- Kuszmaul, W. (2018). Fast algorithms for finding pattern avoiders and counting pattern occurrences in permutations. Mathematics of Computation, 87(310), 987-1011.
- Mansour, T., Vainshtein, A. (2001). Restricted 132-avoiding permutations. Advances in Applied Mathematics, 26(3), 258-269.
- The On-Line Encyclopedia of Integer Sequences (OEIS)
- Simion, R., Schmidt, F. W. (1985). Restricted permutations. European Journal of Combinatorics, 6(4), 383-406.