King Permutations on the Cylinder

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Cylindrical chess game

Consider a chess game in which each piece can move off its column and reappear at the beginning of that column.

This is what we call a cylindrical chess board.
Cylindrical chess game

In how many ways can we place \( n \) kings on a cylindrical board such that:

❖ Each row and each column contains exactly one king.
❖ The kings are not attacking each other.

Want to try? On a 5*5 cylindrical board, put five kings
http://aviad.hagitbagno.com/fivekings/index.html
Definition: Cylindrical King Permutation

A permutation $\sigma = [\sigma_1, ..., \sigma_n] \in S_n$ is called a \textit{cylindrical king permutation} if $|\sigma_i - \sigma_{i+1}| > 1$ for each $1 \leq i \leq n - 1$ and $|\sigma_1 - \sigma_n| > 1$. 
**Definition: Bond and cyclic bond**

Let $\sigma = [\sigma_1, ..., \sigma_n] \in S_n$ and let $i \in [n-1]$.

- We say that the pair $(\sigma_i, \sigma_{i+1})$ is a *(regular) bond* in $\sigma$ if $|\sigma_i - \sigma_{i+1}| = 1$.
- If $|\sigma_1 - \sigma_n| = 1$ then we say that the pair $(\sigma_n, \sigma_1)$ is an *edge bond* in $\sigma$.
- In general, adopting the convention that $\sigma_{n+1} = \sigma_1$, we say that $(\sigma_i, \sigma_{i+1})$ is a *cyclic bond* if it is a regular or an edge bond.
**Example:** Bond and cyclic bond

In $\sigma = [41325] \in S_5$ there are 2 cyclic bonds.

The **regular bond** (3,2) and the **edge bond** (5,4)
Cylindrical King Permutations

- According to this new definition, a permutation is a cylindrical king if and only if it has no cyclic bonds.
- This type of modification from regular to cyclic parameters has been used also in the case of the descent parameter. (See for example in: https://link.springer.com/content/pdf/10.1007/s10801-005-4532-5.pdf)
- Aside from its role as identifying the cylindrical kings, the definition of cyclic bonds leads to some interesting counting results by itself.
Distribution of the number of regular bonds and cyclic bonds.

For each $\sigma \in S_n$ we denote by $bnd(\sigma)$ the number of regular bonds in $\sigma$ and by $cbnd(\sigma)$ the number of cyclic bonds in $\sigma$. 
Distribution of the number of regular bonds and cyclic bonds

The generating functions:

\[ B_n(t) = \sum_{\sigma \in S_n} t^{bnd(\sigma)}, \quad CB_n(t) = \sum_{\sigma \in S_n} t^{cbnd(\sigma)}. \]

Thus:

\[ B_1(t) = CB_1(t) = 1 \]
\[ B_2(t) = 2t, \quad CB_2(t) = 2t^2 \]
\[ B_3(t) = 2t^2 + 4t, \quad CB_3(t) = 6t^2. \]
Theorem Distribution of the number of cyclic bonds.

For $n \geq 2$:

$$CB_{n+1}(t) = (n + 1)B_n(t) + 2(n + 1) \sum_{i=1}^{n} (t - 1)^i B_{n-i}(t)$$
**Theorem** Distribution of the number of regular bonds.

For $n \geq 1$:

$$B_n(t) = CB_n(t) + \frac{1}{n} (1 - t)CB'_n(t).$$
Theorem Asymptotic value: Cylindrical King Permutations $CK_n$.

For $n > 2$:

- $|A_n| = 2|K_{n-1}| + |A_{n-2}|$, where $A_n = K_n - CK_n$.

- The asymptotic value of $\frac{|CK_n|}{|K_n|}$ is equal to 1.
The Poset $\mathbb{C}\mathbb{K} = \bigcup_{n \geq 1} C K_n$

- For $\sigma \in S_n$ and $\pi \in S_k$ ($k < n$), we say that $\sigma$ *contains* $\pi$ if there is a subsequence of (the one line notation of) $\sigma$ that is order-isomorphic to that of $\pi$.

- We write $\pi \prec \sigma$ and study the sub-poset $\mathbb{C}\mathbb{K} = \bigcup_{n \geq 1} C K_n$ of the poset $\mathbb{K} = \bigcup_{n \geq 1} K_n$ with respect to the containment relation.
Example: The Poset $\mathbb{C} \mathbb{K} = \bigcup_{n \geq 1} C K_n$. 
**Theorem** The maximal gap between two permutations in $\mathbb{CK}$ is 4

Let $\sigma, \pi \in \mathbb{CK}$ be such that $\pi < \sigma$ and $|\sigma| - |\pi| > 4$. Then there is some $\tau \in \mathbb{CK}$ such that $\pi < \tau < \sigma$ and $|\sigma| - |\tau| \leq 4$. 
**Definition**: The cyclic path and cyclic length.

For $1 \leq i < j \leq n$, we denote by $((i, j))$ the shorter path on the circle leading from $i$ to $j$ whose length is the minimum of $j - i$ and $n - j + i$.

We also set $||j - i|| = |((i, j))|$, i.e. $||j - i||$ is the length of the shorter path from $i$ to $j$. 
The cyclic path and cyclic length.

\[ \sigma = [26415837] \quad |2 - 7| = 3 \]
Definition: The cyclic Manhattan distance

Let $\sigma \in S_n$ and let $i < j \in [n]$. Let the cyclic (Manhattan) distance between the $i$–th and $j$–th entries be defined as:

$$cd_{\sigma}(i, j) = |\sigma_j - \sigma_i| + |j - i|$$

The **cyclic breadth** of $\sigma \in S_n$ is then defined to be:

$$cbr(\sigma) = \min_{i,j\in[n],i\neq j} cd_{\sigma}(i, j)$$
**Example:** The cyclic Manhattan distance.

Let $\sigma = [724915836] \in S_9$.

Then $cd_\sigma(1,2) = (7 - 2) + |2 - 1| = 6,$

$cd_\sigma(2,5) = (2 - 1) + |5 - 2| = 4,$ and

$cd_\sigma(1,9) = (7 - 6) + |9 - 1| = 2.$

❖ The breath of $\sigma$ is $cbr(\sigma) = 2$
Theorem Descendants.

Omitting a single entry from a permutation may decrease the cyclic breadth by at most one.

* Note that omitting an element $\sigma_k$ might also increase the breadth of a permutation $\sigma$. 
Theorem Descendants.

Let $\sigma \in CK_n$.

There are $n$ distinct permutations $\sigma' \in CK_{n-1}$ such that $\sigma' \prec \sigma$ if and only if $cbr(\sigma) > 3$.
Thank you for your attention

Eli, Shulamit, Moriah and Estrella.

For more reading:
https://www.researchgate.net/publication/338500316_Counting_King_Permutations_on_the_Cylinder