# Inversion sequences avoiding pairs of patterns

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## Introduction

The enumeration of inversion sequences avoiding a single pattern was initiated by Corteel–Martinez–Savage–Weselcouch and Mansour–Shattuck independently. Their work has sparked various investigations of generalized patterns in inversion sequences, including patterns of relation triples by Martinez and Savage (2018), consecutive patterns by Auli and Elizalde (2019), and vincular patterns by Lin and Yan (2020).

**Definition 1.** *A permutation of $[n] := \{1, 2, \ldots, n\}$ is a word $\pi_1\pi_2\cdots\pi_n$ such that $\{\pi_1, \pi_2, \ldots, \pi_n\} = [n]$. The set of permutations of $[n]$ is denoted by $\mathfrak{S}_n$.*

**Definition 2.** *An inversion sequence of length $n$ is an integer sequence $(e_1, e_2, \ldots, e_n)$ with the restriction $0 \le e_i \le i - 1$ for all $1 \le i \le n$. The set of inversion sequences of length $n$ is denoted by $\mathbf{I}_n$.*

The $n$-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ enumerates Dyck paths of order $n$.

**Theorem 1.** *For any $n \ge 1$ and a pattern pair $p \in \{(011, 021), (010, 021)\}$, we have $|\mathbf{I}_n(p)| = C_n$.*

**Remark 1.** *Theorem 1 was proved by Martinez and Savage, but we find a bijective proof.*

The Fibonacci number $F_n$ can be defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$ and the initial values $F_0 = 0$ and $F_1 = 1$.

**Theorem 2.** *For any $n \ge 1$ and a pattern pair $p \in \{(012, 102), (012, 120), (011, 102)\}$, we have $|\mathbf{I}_n(p)| = F_{2n-1}$.*

Martinez and Savage showed that $|\mathbf{I}_n(012)| = F_{2n-1}$ and we observe that $\mathbf{I}_n(012) = \mathbf{I}_n(012, 102) = \mathbf{I}_n(012, 120)$. We construct a recursive bijection between $\mathbf{I}_n(012)$ and $\mathbf{I}_n(011, 102)$.

**Question 1.** *The number of $(321, 3412)$-avoiding permutations, also known as Boolean permutations, in $\mathfrak{S}_n$ is $F_{2n-1}$. Can one construct a bijection between $\mathfrak{S}_n(321, 3412)$ and $\mathbf{I}_n(012)$ or $\mathbf{I}_n(011, 102)$?*

**Definition 3.** *A word $W = w_1w_2\cdots w_n$ is said to avoid the word (or* pattern*) $P = p_1p_2\cdots p_k$ ($k \le n$) if there does not exist $i_1 < i_2 < \cdots < i_k$ such that the subword $w_{i_1}w_{i_2}\cdots w_{i_k}$ of $W$ is order isomorphic to $P$. For a set of words $\mathcal{W}$ and a sequence of patterns $P_1, \ldots, P_r$, let us denote $\mathcal{W}(P_1, \ldots, P_r)$ the set of words in $\mathcal{W}$ which avoid all patterns $P_i$ for $i = 1, \ldots, r$.*

**Definition 4.** *Fix a triple of binary relations $(\rho_1, \rho_2, \rho_3) \in \{<, >, \le, \ge, =, \ne, -\}^3$. Let $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ be the set of sequences $e \in \mathbf{I}_n$ with no $i < j < k$ such that*

$$e_i\, \rho_1\, e_j, \quad e_j\, \rho_2\, e_k \quad \text{and} \quad e_i\, \rho_3\, e_k.$$

*Here the relation "$-$" on a set $S$ is all of $S \times S$, i.e., $x$ "$-$" $y$ for all $x, y \in S$. For example, we have $\mathbf{I}_n(<, -, <) = \mathbf{I}_n(012, 021, 011)$ and $\mathbf{I}_n(<, >, \ne) = \mathbf{I}_n(021, 120)$.*

The systematic study of the enumerative aspect of inversion sequences avoiding triple of binary relations was initialed by Martinez and Savage (2018). They proved that $|\mathbf{I}_n(>, \ne, -)| = 1 + \sum_{i=1}^{n-1}\binom{2i}{i-1}$ and posed the following conjecture.

**Conjecture 3** (Martinez and Savage [3]). *For $n \ge 1$, we have*

$$|\mathbf{I}_n(021, 120)| = 1 + \sum_{i=1}^{n-1}\binom{2i}{i-1}.$$

This work was motivated by their conjecture.

## The purpose of this work

In this poster, we carried out the systematic study of inversion sequences avoiding two patterns of length 3. Our enumerative results establish further connections to the OEIS sequences and some classical combinatorial objects, such as restricted permutations, weighted ordered trees and set partitions. Since patterns of relation triples are some special multiple patterns of length 3, our results complement the work by Martinez and Savage. In particular, we provide an algebraic proof of Conjecture 3.

## Pattern pair (101,021): A bijection to weighted ordered trees

The algebraic generating function

$$A(t) = 1 + \frac{tA(t)}{1 - tA(t)^2}$$

appearing as A106228 in the OEIS enumerates many interesting combinatorial objects, including

- $(4123, 4132, 4213)$-avoiding permutations, proved by Albert et al. [1];
- $(101, 102)$-avoiding inversion sequences, proved by Cao et al. [2];
- weighted ordered trees, where each interior vertex (non-root, non-leaf) is weighted by a positive integer less than or equal to its outdegree.

We construct a "type"-preserving bijection from ordered trees of $n$ edges to Dyck paths of length $n$ and prove a new interpretation for A106228.

**Theorem 4.** *Inversion sequences avoiding the pair $(101, 021)$ are counted by the integer sequence A106228.*

**Definition 5.** *Let $\mathbf{t} = (t_1, t_2, \ldots, t_k)$ be a sequence of positive integers with $\sum_i t_i = n$. A Dyck path $D = h_1h_2\cdots h_n$ ($h_i$ is the height of $i$th east step) has type $\mathbf{t}$ if*

$$h_i \ne h_{i+1} \Longleftrightarrow i \in \{t_1, t_1 + t_2, \ldots, t_1 + t_2 + \cdots + t_{k-1}\}.$$

*For example, the Dyck path in the right-bottom side of Fig. 1 has type $(3, 1, 2, 1)$. Let $\mathfrak{D}_n$ be the set of Dyck paths of length $n$.*

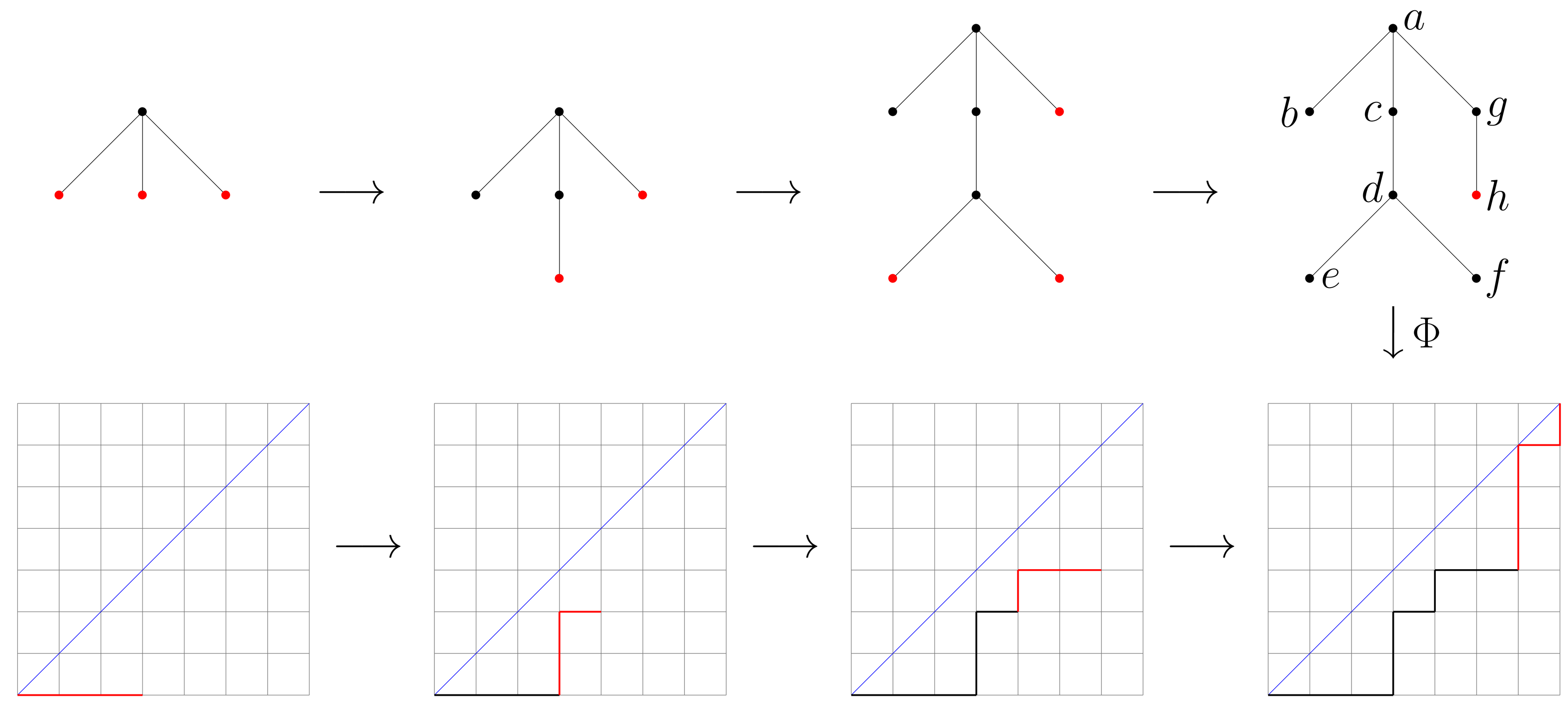


Figure 1: An example of the construction of the bijection $\Phi$

**Definition 6.** *In an ordered tree (or plane tree) $T$, the outdegree of a vertex $v \in V(T)$, denoted $\deg_v$, is the number of descendants of $v$. All vertices of $T$ are ordered by the preorder (or depth-first order). If $v_1$ is the root of $T$ and $v_2, v_3, \ldots, v_k$ are the interior vertices of $T$ in preorder, then we called $T$ has type $(\deg_{v_1}, \deg_{v_2}, \ldots, \deg_{v_k})$. For instance, the tree in our running example has type $(\deg_a, \deg_c, \deg_d, \deg_g) = (3, 1, 2, 1)$. Let $\mathfrak{T}_n$ be the set of all ordered trees with $n$ edges.*

**Theorem 5.** *There exists a type-preserving bijection $\Phi$ from $\mathfrak{T}_n$ to $\mathfrak{D}_n$.*

**References**

[1] M.H. Albert, C. Homberger, J. Pantone, N. Shar and V. Vatter, Generating permutations with restricted containers, J. Combin. Theory Ser. A, **157** (2018), 205–232.

[2] W. Cao, E.Y. Jin and Z. Lin, Enumeration of inversion sequences avoiding triples of relations, Discrete Appl. Math., **260** (2019), 86–97.

[3] M. Martinez and C.D. Savage, Patterns in inversion sequences II: Inversion sequences avoiding triples of relations, J. Integer Seq., **21** (2018), Article 18.2.2.

## Pattern pairs $(110,102)$ and $(120,102)$

For a Dyck path $D = h_1 \cdots h_n \in \mathfrak{D}_n$, let $last(D) = h_n$ be the height of the last east step of $D$. Let us define $d_{n,m}$ the number of Dyck paths $D \in \mathfrak{D}_n$ with $last(D) = m$. Let

$$D(u,x) := \sum_{n\geq 1} d_{n,m} u^m x^n = x + (1+u)x^2 + (2u^2+2u+1)x^3 + \cdots$$

be the enumerator of Dyck paths by the height of their last steps. It is known that $D(u,x)$ is algebraic and

$$D(u,x) = \frac{2x}{1-2x+\sqrt{1-4ux}}.$$

For any $0 \leq m < \ell \leq n$, denote by $a_{n,m,\ell}$ the number of $(110,102)$-avoiding inversion sequences of length $n$ in which the largest entry is $m$ with the left-most occurrence of $m$ in position $\ell$. Then, the first few values of the arrays $[a_{n,m,\ell}]_{0\leq m<\ell\leq n}$ are:

$$\begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&2&1\\0&0&2\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&3&2&1\\0&0&6&3\\0&0&0&5\end{bmatrix}, \begin{bmatrix}1&0&0&0&0\\0&4&3&2&1\\0&0&12&8&4\\0&0&0&19&9\\0&0&0&0&14\end{bmatrix}.$$

We have the following recurrence relation for $a_{n,m,\ell}$.

**Lemma 1.** *For $1 \leq m < \ell \leq n$, we have*

$$a_{n,m,\ell} = d_{n-1,m} - d_{\ell-1,m} + d_{\ell-1,m-1} - \chi(m=1,\ell=2) + \sum_{j=0}^{m}\sum_{k=\ell-1}^{n-1} a_{n-1,j,k},$$

*where $a_{n,0,\ell} = \delta_{\ell,1}$ for $1 \leq \ell \leq n$.*

For any $n \geq 2$, let

$$a_n(u,v) := \sum_{\ell=2}^{n}\sum_{m=1}^{\ell-1} a_{n,m,\ell} u^m v^\ell.$$

Introduce the generating function $a(x;u,v)$ by

$$a(x;u,v) := \sum_{n\geq 2} a_n(u,v)x^n = uv^2x^2 + (2uv^2+uv^3+2u^2v^3)x^3 + \cdots.$$

**Lemma 2.** *The functional equation for $a(x;u,v)$ is*

$$a(x;u,v) = \frac{vx^2}{x-1} + \frac{vx}{1-v}D(uv,x) + \left(\frac{uvx-vx}{1-x} - \frac{v^2x}{1-v}\right)D(u,xv) + \\ + \frac{uv^2x}{(1-v)(1-uv)}a(x;uv,1) + \frac{u^2v^2x}{(1-u)(1-uv)}a(x;1,uv) - \frac{uv^2x}{(1-u)(1-v)}a(x;u,v).$$

**Theorem 3.** *We have*

$$a(x;1,1) = \frac{x^2 + x^2\sqrt{1-4x}}{(x-1)((3x-1)\sqrt{1-4x} - 4x^2 + 5x - 1)}.$$

*Equivalently,*

$$|I_n(110,102)| = 1 + \sum_{i=1}^{n-1}\binom{2i}{i-1}.$$

Similar approach can be applied to prove:

**Theorem 4.** *For $n \geq 1$, we have*

$$|\mathbf{I}_n(120,102)| = 1 + \sum_{i=1}^{n-1}\binom{2i}{i-1}.$$

## Pattern pair (120,021): Martinez–Savage conjecture

The generating function for Catalan numbers is

$$C(x) := 1 + \sum_{n\geq 1} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

For our purpose, we color each east step of a Dyck path by black or red and call such a Dyck path a *colored Dyck path*. If the $i$-th east step has height $k$ and color red, then we write $h_i = \bar{k}$. It was observed in [1] that an inversion sequence is $021$-avoiding if and only if its positive entries are weakly increasing. By this characterization, each sequence $e \in \mathbf{I}_n(021)$ can be represented by a colored Dyck path $\mathcal{H}(e) = h_1h_2\cdots h_n$, called the *outline of* $e$ in [2, Def. 2.1], where

$$h_i = \begin{cases} e_i, & \text{if } e_i > 0, \\ \bar{k}, & \text{if } e_i = 0 \text{ and } k = \max\{e_1,\ldots,e_i\}. \end{cases}$$

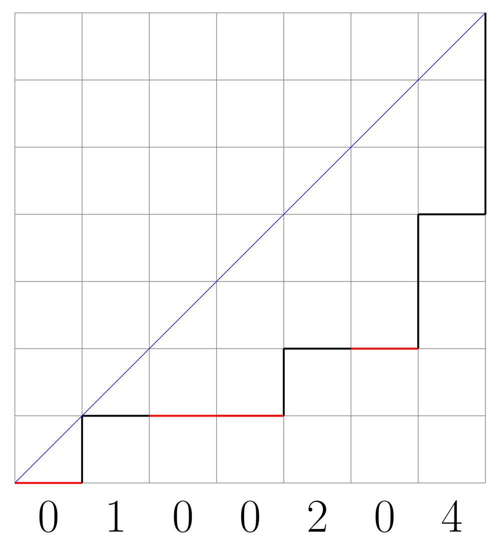


Figure 1: The outline of an inversion sequence

Let us denote by $\mathcal{A}_n$ the set of colored Dyck paths of length $n$ satisfying

- all the east step of height $0$ are colored red, and
- the first east step of each positive height is colored black.

It clear that the geometric representation $\mathcal{H}$ is a bijection between $\mathbf{I}_n(021)$ and $\mathcal{A}_n$. Let $\mathcal{B}_n$ be the set of colored Dyck paths in $\mathcal{A}_n$ such that only east steps with smallest positive height can have two colors. The colored Dyck path $\bar{0}1\bar{1}\bar{1}2\bar{2}4$ in Fig. 1 is an element in $\mathcal{A}_7 \setminus \mathcal{B}_7$. Note that $I_n(<,>,\neq) = I_n(021,120)$ and the geometric representation $\mathcal{H}$ induces a bijection between $\mathbf{I}_n(021,120)$ and $\mathcal{B}_n$. Let

$$S(x) := 1 + \sum_{n\geq 1} |\mathcal{B}_n| x^n = 1 + x + 2x^2 + 6x^3 + 21x^4 + \cdots.$$

**Lemma 5.** *We have the following functional equation for $S(x)$:*

$$S(x) = 1 + xH(x)/(1-x) + xC(x)(S(x) - 1/(1-x)).$$

Let $\mathcal{C}_n$ be the set of colored Dyck paths of length $n$ such that only east steps with height $0$, except the first step (which is colored red), can have two colors. Introduce the generating function

$$H(x) := 1 + \sum_{n\geq 0} |\mathcal{C}_n| x^n = 1 + x + 3x^2 + 10x^3 + 35x^4 + \cdots.$$

**Lemma 6.** *We have the following functional equation for $H(x)$:*

$$H(x) = 1 + x(2H(x) - 1)C(x). \tag{1}$$

**Theorem 7.** *We have*

$$S(x) = \frac{1 - 4x + \sqrt{-16x^3 + 20x^2 - 8x + 1}}{2(x-1)(4x-1)}.$$

*Consequently, the Martinez–Savage conjecture (Conjecture 3) is true.*

## Wilf-equivalences

**Theorem 8.** *The following three Wilf-equivalences hold:*

$$(011,201) \sim (011,210), \quad (000,201) \sim (000,210) \quad \textit{and} \quad (100,021) \sim (110,021).$$

The proof uses a bijection $\varphi : \mathbf{I}_n(210) \to \mathbf{I}_n(201)$ of Corteel et al. [1, Thm. 5].

**Theorem 9.** *There exists a bijection $\psi : \mathbf{I}_n(201) \to \mathbf{I}_n(210)$ which preserves the pattern $010$ and the triple of statistics $(\mathrm{zero}, \mathrm{dist}, \mathrm{satu})$. In particular, $(010,201) \sim (010,210)$.*

The bijection $\varphi : \mathbf{I}_n(210) \to \mathbf{I}_n(201)$ of Corteel et al. [1, Thm. 5] does not preserves the pattern $010$.

## Conjectures

**Conjecture 10.** *The following Wilf-equivalences hold:*

$$(011,201) \sim (011,210) \sim (110,210,120,010) \sim (100,210,120,010).$$

**Conjecture 11** (Lin and Ma 2019). *For $n \geq 1$, we have*

$$|I_n(0012)| = 1 + \sum_{i=1}^{n-1}\binom{2i}{i-1}.$$

### References


[1] S. Corteel, M. Martinez, C.D. Savage and M. Weselcouch, Patterns in inversion sequences I, Discrete Math. Theor. Comput. Sci., **18** (2016), #2.

[2] Z. Lin and D. Kim, A sextuple equidistribution arising in pattern avoidance, J. Combin. Theory Ser. A, **155** (2018), 267–286.

## Pattern pairs whose avoidance sequences appear to match sequences in the OEIS

| Pattern pair $p$ | $a_n = \| I_n(p)\|$ counted by: | solved? | OEIS | $a_8$, equiv class |
|---|---|---|---|---|
| $(001, 010)$ | $n$ | yes | A000027 | 8,A |
| $(001, 011)$ | $n$ | yes | A000027 | 8,B |
| $(001, 012)$ | $n$ | yes | A000027 | 8,C |
| $(001, 110)$ | Lazy caterer sequence | yes | A000124 | 29,A |
| $(001, 021)$ | Lazy caterer sequence | yes | A000124 | 29,B |
| $(001, 120)$ | Lazy caterer sequence | yes | A000124 | 29,C |
| $(000, 001)$ | Fibonacci number $F_{n+1}$ | yes | A000045 | 34 |
| $(001, 100)$ | $F_{n+2} - 1$ | yes | A000071 | 54 |
| $(001, 210)$ | Cake number $\binom{n}{3} + n$ | yes | A000125 | 64 |
| $(000, 011)$ | $2^{n-1}$ | yes | A000079 | 128,A |
| $(001, 101)$ | $2^{n-1}$ | yes | A000079 | 128,B |
| $(001, 102)$ | $2^{n-1}$ | yes | A000079 | 128,C |
| $(001, 201)$ | $2^{n-1}$ | yes | A000079 | 128,D |
| $(010, 012)$ | $2^{n-1}$ | yes | A000079 | 128,E |
| $(011, 012)$ | $2^{n-1}$ | yes | A000079 | 128,F |
| $(110, 012)$ | $2^n - n$ | yes | A000325 | 248,A |
| $(012, 021)$ | $2^n - n$ | yes | A000325 | 248,B |
| $(012, 201)$ | $\|\mathfrak{S}_n(321, 2143)\|$ | yes | A088921 | 411,A |
| $(012, 210)$ | $\|\mathfrak{S}_n(321, 2143)\|$ | yes | A088921 | 411,B |
| $(011, 102)$ | $F_{2n-1}$ | yes | A001519 | 610,A |
| $(012, 102)$ | $F_{2n-1}$ | yes | A001519 | 610,B |
| $(012, 120)$ | $F_{2n-1}$ | yes | A001519 | 610,C |
| $(010, 011)$ | $\sum_{k=0}^{n-1}(n-k)^k$ | yes | A026898 | 733 |
| $(011, 021)$ | Catalan number $C_n$ | yes | A000108 | 1430,A |
| $(010, 021)$ | Catalan number $C_n$ | yes | A000108 | 1430,B |
| $(011, 201)$ | $\| I_n(-, >, \geq)\| = \| I_n(\neq, \geq, \geq)\|$ | Open | A279555 | 3091,A |
| $(011, 210)$ | $\| I_n(-, >, \geq)\| = \| I_n(\neq, \geq, \geq)\|$ | Open | A279555 | 3091,B |
| $(021, 120)$ | $1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}$ | yes | A279561 | 4082,A |
| $(102, 120)$ | $1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}$ | yes | A279561 | 4082,B |
| $(110, 102)$ | $1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}$ | yes | A279561 | 4082,C |
| $(010, 100)$ | Bell number $B_n$ | yes | A000110 | 4140,D |
| $(011, 101)$ | Bell number $B_n$ | yes | A000110 | 4140,E |
| $(011, 110)$ | Bell number $B_n$ | yes | A000110 | 4140,F |
| $(101, 021)$ | $\|\mathfrak{S}_n(4123, 4132, 4213)\|$ | yes | A106228 | 5798,B |
| $(021, 201)$ | Large Schröder number $S_n$ | yes | A006318 | 8558,A |
| $(021, 210)$ | Large Schröder number $S_n$ | yes | A006318 | 8558,B |
| $(101, 110)$ | indecompsable set partitions | yes | A074664 | 11624 |

## References


[1] M. Martinez and C.D. Savage, Patterns in inversion sequences II: Inversion sequences avoiding triples of relations, J. Integer Seq., **21** (2018), Article 18.2.2.

[2] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, http://oeis.org, 2019.

Pattern pairs whose avoidance sequences either have been studied in [1, 2, 3] or appear to be new in the OEIS

| Pattern pair $p$ | $a_n = \lvert\mathbf{I}_n(p)\rvert$ counted by: | calculated? | in OEIS? | $a_8$, equiv class |
|---|---|---|---|---|
| $(000, 012)$ | $1, 2, 4, 5, 21, 0, 0, \ldots$ | Ultimately zero | new | 0 |
| $(100, 012)$ | $1, 2, 5, 12, 27, 56, 110, \ldots$ | Open | New | 207 |
| $(101, 012)$ | $\lvert\mathfrak{S}_n(321, 2413, 3142)\rvert$ | In [2, Sec. 2.8] | A034943 | 351 |
| $(000, 021)$ | $1, 2, 5, 14, 39, 111, 317, \ldots$ | Open | New | 911 |
| $(000, 102)$ | $1, 2, 5, 14, 40, 121, 373, \ldots$ | Open | New | 1181 |
| $(000, 010)$ | $1, 2, 4, 10, 29, 95, 345, \ldots$ | Open | A279552 | 1376 |
| $(011, 120)$ | $1, 2, 5, 14, 42, 132, 431, \ldots$ | Open | New | 1452 |
| $(100, 011)$ | Nexus numbers | In [2, Sec. 2.15] | A047970 | 2048 |
| $(021, 102)$ | $1, 2, 6, 20, 66, 213, 683, \ldots$ | Open | New | 2211 |
| $(010, 102)$ | $1, 2, 5, 15, 51, 186, 707, \ldots$ | Open | New | 2763 |
| $(000, 120)$ | $1, 2, 5, 15, 50, 185, 737, \ldots$ | Open | New | 3126 |
| $(010, 120)$ | $1, 2, 5, 15, 52, 201, 845, \ldots$ | Open | A279559 | 3801 |
| $(010, 110)$ | $1, 2, 5, 15, 52, 201, 847, \ldots$ | Open | New | 3836 |
| $(010, 101)$ | Bell number $B_n$ | In [2, Sec. 2.17] | A000110 | 4140,A |
| $(000, 101)$ | Bell number $B_n$ | In [3, Sec. 6] | A000110 | 4140,B |
| $(000, 110)$ | Bell number $B_n$ | In [3, Sec. 6] | A000110 | 4140,C |
| $(100, 021)$ | $1, 2, 6, 21, 78, 297, 1144, \ldots$ | Open | New | 4433,A |
| $(110, 021)$ | $1, 2, 6, 21, 78, 297, 1144, \ldots$ | Open | New | 4433,B |
| $(010, 201)$ | $1, 2, 5, 15, 53, 214, 958, \ldots$ | Open | New | 4650,A |
| $(010, 210)$ | $1, 2, 5, 15, 53, 214, 958, \ldots$ | Open | New | 4650,B |
| $(100, 102)$ | $1, 2, 6, 21, 80, 318, 1305, \ldots$ | Open | New | 5487 |
| $(102, 210)$ | $1, 2, 6, 22, 87, 351, 1416, \ldots$ | Open | New | 5681 |
| $(101, 102)$ | $\lvert\mathfrak{S}_n(4123, 4132, 4213)\rvert$ | In [3, Sec. 4] | A106228 | 5798,A |
| $(102, 201)$ | $1, 2, 6, 22, 87, 354, 1465, \ldots$ | Open | A279566 | 6154 |
| $(000, 201)$ | $1, 2, 5, 16, 60, 257, 1218, \ldots$ | Open | New | 6270,A |
| $(000, 210)$ | $1, 2, 5, 16, 60, 257, 1218, \ldots$ | Open | New | 6270,B |
| $(000, 100)$ | $1, 2, 5, 16, 60, 260, 1267, \ldots$ | Open | A279564 | 6850 |
| $(101, 120)$ | $1, 2, 6, 22, 90, 397, 1859, \ldots$ | Open | New | 9145 |
| $(100, 120)$ | $1, 2, 6, 22, 92, 421, 2062, \ldots$ | Open | New | 10646 |
| $(110, 120)$ | $1, 2, 6, 22, 92, 423, 2091, \ldots$ | Open | A279570 | 10950 |
| $(100, 110)$ | $1, 2, 6, 22, 93, 437, 2233, \ldots$ | Open | New | 12227 |
| $(100, 101)$ | $1, 2, 6, 22, 93, 439, 2267, \ldots$ | Open | New | 12628 |
| $(120, 201)$ | $1, 2, 6, 23, 102, 498, 2607, \ldots$ | Open | New | 14386 |
| $(120, 210)$ | $1, 2, 6, 23, 102, 499, 2625, \ldots$ | Open | A279573 | 14601 |
| $(110, 201)$ | $1, 2, 6, 23, 103, 512, 2739, \ldots$ | Open | New | 15464 |
| $(101, 210)$ | $1, 2, 6, 23, 103, 513, 2763, \ldots$ | Open | New | 15816 |
| $(101, 201)$ | $\lvert\mathfrak{S}_n(21\bar{3}54)\rvert$ | In [2, Sec. 2.27] | A117106 | 17734,A |
| $(100, 210)$ | $\lvert\mathfrak{S}_n(21\bar{3}54)\rvert$ | In [1, Sec. 3.3] | A117106 | 17734,B |
| $(100, 201)$ | $\lvert\mathfrak{S}_n(21\bar{3}54)\rvert$ | In [2, Sec. 2.27] | A117106 | 17734,C |
| $(110, 210)$ | $\lvert\mathfrak{S}_n(21\bar{3}54)\rvert$ | In [2, Sec. 2.27] | A117106 | 17734,D |
| $(201, 210)$ | $\lvert\mathfrak{S}_n(MMP(0, 2, 0, 2))\rvert$ | In [2, Sec. 2.27] | A212198 | 23072 |

## References


[1] M. Bouvel, V. Guerrini, A. Rechnitzer, and S. Rinaldi, Semi-Baxter and strong-Baxter: two relatives of the Baxter sequence, SIAM J. Discrete Math., **32(4)** (2018), 2795–2819

[2] M. Martinez and C.D. Savage, Patterns in inversion sequences II: Inversion sequences avoiding triples of relations, J. Integer Seq., **21** (2018), Article 18.2.2.

[3] W. Cao, E.Y. Jin and Z. Lin, Enumeration of inversion sequences avoiding triples of relations, Discrete Appl. Math., **260** (2019), 86–97.

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