mod k-alternating permutations in $\mathcal{S}_n(132)$

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2020 Permutation Patterns Virtual Workshop

Outline

Motivation

Ran Pan's Project P *Project* P

http://www.math.ucsd.edu/∼projectp/

Problem 10: enumerate parity-alternating permutations in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$.

Then we conducted researches on mod k-alternating permutations in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$.

Outline

Permutations

- The set of permutations of $[n]$ is denoted by $\mathcal{S}_n$.
- We let $red(w)$ denote the permutation of $[n]$ obtained from w by replacing the i-th smallest letter in w by i, e.g. $red(4592) = 2341$.
- We let $\mathcal{S}_n(\lambda)$ denote the set of permutations in $\mathcal{S}_n$ avoiding λ.
- $\left|\mathcal{S}_n(132)\right| = \left|\mathcal{S}_n(123)\right| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number.

Parity-alternating permutations

- Given $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{S}_n$, we say σ is parity-alternating (or $\bmod 2$ alternating) if $\sigma_i \equiv a + i \pmod 2$ for some integer a, i.e. σ is odd–even alternating or even–odd alternating.
- Let $(m)_k$ denote m modulo k, and let $(\sigma)_k = ((\sigma_1)_k \ldots (\sigma_n)_k)$, then σ is parity-alternating if $(\sigma)_2 = (0101\ldots)$ or $(1010\ldots)$.
- Ex. $\sigma = 256341$ and $\tau = 3672145$ are parity-alternating.

mod k-alternating permutations

- Given $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{S}_n$, we say σ is mod k-alternating if $\sigma_i \equiv a + i \pmod{k}$ for some a.

- σ is mod k-alternating if $(\sigma)_k$ is one of $(12\ldots(k-1)012\ldots(k-1)\ldots)$, $(23\ldots0123\ldots01\ldots)$, $\ldots$, or $(01\ldots(k-2)(k-1)01\ldots(k-2)(k-1)\ldots)$.

- Ex. $\sigma = 264531$ and $\tau = 1564237$ are mod 3-alternating, because $(\sigma)_3 = (201201)$, $(\tau)_3 = (1201201)$.

mod k-alternating permutations — Notations

- $\mathcal{AS}_n^k = \{\sigma \in \mathcal{S}_n : \sigma \text{ is mod } k\text{-alternating}\}$
- $\mathcal{AS}_{n,j}^k = \{\sigma \in \mathcal{S}_n : \sigma \text{ is mod } k\text{-alternating}, (\sigma_1)_k = j\}$

- $|\mathcal{AS}_{2n}^2| = 2(n!)^2, \qquad |\mathcal{AS}_{2n+1}^2| = n!(n+1)!.$
- $|\mathcal{AS}_{nk}^k| = k(n!)^k, \qquad |\mathcal{AS}_{nk+r}^k| = ((n+1)!)^r(n!)^{(k-r)}$

Our problem: enumerating mod k-alternating permutations in $\mathcal{S}_n(\tau)$

- $\mathcal{AS}_n^k(\tau) = \mathcal{AS}_n^k \cap \mathcal{S}_n(\tau)$

- $\mathcal{AS}_{n,j}^k(\tau) = \mathcal{AS}_{n,j}^k \cap \mathcal{S}_n(\tau)$

- Main goal: enumerate $|\mathcal{AS}_n^k(\tau)|$ and $|\mathcal{AS}_{n,j}^k(\tau)|$.

- Especially, we study the case when $\tau = 123, 132, 312$ and 321 is of length 3.

Generating Functions

- Results about $\tau = 123$ and 321 are mostly open. Results about $\tau = 312$ are (almost) trivial.
- When $\tau = 132$, let $a_n^{(k)} = |\mathcal{AS}_n^k(132)|$ and $a_{n,j}^{(k)} = |\mathcal{AS}_{n,j}^k(132)|$.

$$A_k(t) = \sum_{n \geq 0} a_n^{(k)} t^n.$$

$$A_{k;i,j}(t) = \sum_{n \geq 0} a_{nk+i,j}^{(k)} t^{nk+i}.$$

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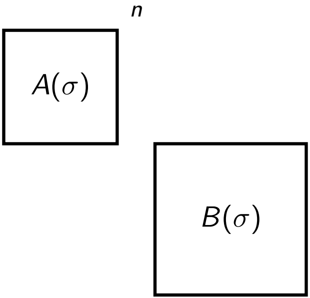
The set $\mathcal{AS}_n^2(132)$

We first consider the set of parity-alternating permutations in $\mathcal{S}_n(132)$.

Recall: for $\sigma \in \mathcal{S}_n(132)$, $\sigma_k = n$, then
$A(\sigma) = red(\sigma_1 \cdots \sigma_{k-1}) \in \mathcal{S}_{k-1}(132)$ and
$B(\sigma) = red(\sigma_{k+1} \cdots \sigma_n) \in \mathcal{S}_{n-k}(132)$.



Computation of $A_2(t)$ and $A_{2;i,j}(t)$

- $A_{2;0,0}(t)$: even sizes begin with even numbers,
- $A_{2;0,1}(t)$: even sizes begin with odd numbers,
- $A_{2;1,1}(t)$: odd sizes begin with odd numbers.

For example, we have:

$$A_{2;0,0}(t) = 1 + t \cdot A_{2;0,1}(t) \cdot A_{2;1,1}(t).$$

Computation of $A_2(t)$ and $A_{2;i,j}(t)$

Theorem

$$
\begin{aligned}
A_{2;0,0}(t) &= 1 + t^2(A_{2;0,0}(t))^3, && (1)\\
A_{2;0,1}(t) &= 1 + t^2(A_{2;0,1}(t))^3, && (2)\\
A_{2;1,1}(t) &= t \cdot (1 - tA_{2;1,1}(t))^{-2}, \quad \text{and} && (3)
\end{aligned}
$$

$$
\begin{aligned}
A_{2;0,0}\big|_{t^{2n}} &= A_{2;0,1}\big|_{t^{2n}} = \frac{1}{n}\binom{3n}{n-1}, && (4)\\
A_{2;1,1}\big|_{t^{2n+1}} &= \frac{1}{n+1}\binom{3n+1}{n}. && (5)
\end{aligned}
$$

These coincide with OEIS A046646 and OEIS A006013.

Outline

The set $\mathcal{AS}_n^k(132)$

Then we generalize to mod k case.

- $A_{k;m,1}(t)$ enumerates the mod k alternating permutations of sizes $m \pmod{k}$, for $m = 1, \ldots, k-1$, and
- $A_{k;0,m}(t)$ enumerates the mod k alternating permutations whose size is $0 \pmod{k}$ begin with a number which is $m \pmod{k}$, for $m = 0, \ldots, k-1$.

For example,

$$A_{k;m,1}(t) = t \cdot A_{k;m-1,1}(t) \cdot A_{k;0,m+1}(t).$$

Computation of $A_k(t)$ and $A_{k;i,j}(t)$

Theorem

For any positive integer k, the functions $A_{k;0,m}(t)$ for any $m \in \{0, \ldots, k-1\}$ are the same, and

$$A_{k;0,0}(t) = 1 + t^k A_{k;0,0}^{k+1}(t), \qquad (6)$$

$$A_{k;m,1}(t) = t^m A_{k;0,0}^{m+1}(t). \qquad (7)$$

The coefficients (Fuss-Catalan numbers!):

$$A_{k;0,m}(t)|_{t^{nk}} = \frac{1}{n}\binom{(k+1)n}{n-1} \quad \textit{and}$$

$$A_{k;m,1}(t)|_{t^{nk+m}} = \frac{m+1}{nk+m+1}\binom{(k+1)n+m}{n}.$$

The sequences $\{A_{k;0,m}(t)|_{t^{nk}}\}_{n\geq 0} = \{\frac{1}{n}\binom{(k+1)n}{n-1}\}_{n\geq 0}$ appears in OEIS when $k = 1, \ldots, 5$ as A000108, A001764, A002293, A002294, A002295.

Outline

Other results

- When $\tau = 312$, let $b_{n,j}^{(k)} = |\mathcal{AS}_{n,j}^{k}(312)|$,

$$B_{k;i,j}(t) = \sum_{n \geq 0} b_{nk+i,j}^{(k)} t^{nk+i}.$$

  We have

$$B_{k;m,1}(t) = \frac{t^m}{1-t^k}, B_{k;0,0}(t) = \frac{t^k}{1-2t^k}, B_{k;0,m}(t) = 0 \text{ if } m \neq 0.$$

- When $\tau = 123$ or 321, It is hard to write recursive equations.
- We define the set of parity alternating circular permutations $\mathcal{ACS}_n$, and the subset $\mathcal{ACS}_n(\tau)$. We enumerate $|\mathcal{ACS}_n(1324)|$ using a similar technique.

Open problems

- Tracking descents and other statistics in $\mathcal{AS}_n^k(132)$.
- Computing generating functions for $|\mathcal{AS}_n^k(123)|$ and $|\mathcal{AS}_n^k(321)|$.
- Enumerating mod k-alternating permutations in other permutation classes, e.g. separable permutations, $\mathcal{S}_n(\tau)$ where $|\tau| = 4$, etc.
- Seeking for other connections between $\mathcal{AS}_n^k(132)$ and Catalan objects. (Fuss-Catalan $\rightarrow$ rational Catalan combinatorics?)

Thank You!