An expression for the Möbius function, $\mu[\sigma, \pi]$, of the permutation pattern poset based on intervals in π

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The Möbius function

Recall that if σ and π are two elements of a poset, then the Möbius function $\mu[\sigma, \pi]$ is defined as

$$\mu[\sigma, \pi] = \begin{cases} 1 & \text{If } \sigma = \pi \\ -\sum_{\sigma \leq \lambda < \pi} \mu[\sigma, \lambda] & \text{otherwise.} \end{cases}$$

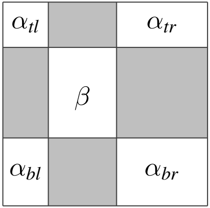
Hall's theorem gives us that

$$\mu[\sigma, \pi] = \sum_{c \in \mathcal{C}} (-1)^{|c|}$$

where $\mathcal{C}$ is the set of chains between σ and π.

Balloon permutations

A *balloon* permutation is formed from the merge of two non-empty permutations α and β, with the property that β occurs as a (possibly trivial) interval.



The relationship between α and β in a balloon permutation.

We write this as $\alpha \circledcirc_{i,j} \beta$, where i, j indicates the position of β in α.

Partitioning the chains

Let σ be any non-empty permutation, and let $\pi = \alpha \odot_{i,j} \beta$, with $\sigma < \pi$.

Our strategy:

- Partition the chains in $[\sigma, \pi]$ into five sets, $\mathcal{B}, \mathcal{S}, \mathcal{R}, \mathcal{M},$ and $\mathcal{P}$.
- Find an expression for the Hall sum of the sets $\mathcal{B}, \mathcal{S}, \mathcal{R},$ and $\mathcal{M}$, either as a constant, or as a sum of values of the Möbius function for a small set of permutations.
- Find a way to ensure that $\mathcal{P}$ is empty.

The last step currently involves adding a restriction to the position of β.

We omit the definitions of the chains from this presentation.

The sets $\mathcal{B}$ and $\mathcal{S}$

We can show that

$$\sum_{c \in \mathcal{B}} (-1)^{|c|} = \begin{cases} \mu[\zeta, \beta] & \text{if } \sigma = \alpha \circledcirc_{i,j} \zeta, \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{S}$ is either empty, or contains exactly one chain of length 1, which occurs when $\sigma = \eta \circledcirc_{i,j} \beta$ for some non-empty η. This trivially gives us

$$\sum_{c \in \mathcal{S}} (-1)^{|c|} = \begin{cases} -1 & \text{if } \sigma \text{ can be written } \eta \circledcirc_{i,j} \beta, \\ 0 & \text{otherwise.} \end{cases}$$

The sets $\mathcal{R}$ and $\mathcal{M}$

The chains in $\mathcal{R}$ are characterised by having the second-highest element be an element of a set of permutations R_π. Permutations in R_π can be written as $\eta \circledcirc_{i,j} \beta$, for some $\eta < \alpha$.

It is easy to show that

$$\sum_{c \in \mathcal{R}} (-1)^{|c|} = \sum_{\lambda \in \mathsf{R}_\pi} \mu[\sigma, \lambda]$$

Finally, we can show that

$$\sum_{c \in \mathcal{M}} (-1)^{|c|} = 0$$

Our main results (part 1)

Our results depend on the form of σ. If $\sigma = \alpha \circledcirc_{i,j} \zeta = \eta \circledcirc_{k,\ell} \beta$, then we will write it in the form $\sigma = \alpha \circledcirc_{i,j} \zeta$.

$$\mu[\sigma,\pi] = \sum_{\lambda \in \mathsf{R}_\pi} \mu[\sigma,\lambda] + \begin{cases} \sum_{c \in \mathcal{P}} (-1)^{|c|} + \mu[\zeta,\beta] & \text{if } \sigma = \alpha \circledcirc_{i,j} \zeta, \\ \sum_{c \in \mathcal{P}} (-1)^{|c|} - 1 & \text{if } \sigma = \eta \circledcirc_{k,\ell} \beta, \\ \sum_{c \in \mathcal{P}} (-1)^{|c|} & \text{otherwise.} \end{cases}$$

The term $\sum_{c \in \mathcal{P}} (-1)^{|c|}$ is not particularly nice, as it is a sum over chains, rather than a sum of Möbius function values.

Our main results (part 2)

If we add the requirement that β is extremal, then we can show that the set $\mathcal{P}$ is empty. This then gives us a nicer set of results.

These are

$$\mu[\sigma,\pi] = \sum_{\lambda \in \mathsf{R}_\pi} \mu[\sigma,\lambda] + \begin{cases} \mu[\zeta,\beta] & \text{if } \sigma = \alpha \circledcirc_{i,j} \zeta, \\ -1 & \text{if } \sigma = \eta \circledcirc_{k,\ell} \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Note that since β can be a trivial interval, we can always find some β that is extremal.