Enumerative combinatorics and pattern avoidance in the matching pattern poset

Matteo Cervetti, Luca Ferrari
matteo.cervetti@unitn.it, luca.ferrari@unifi.it

Synopsis
We investigate the combinatorics of the matching poset pattern avoidance. Juxtaposition of two patterns: enumeration of two classes. Lift-rightmost vertices. The two possible structures for a matching containing minimally containing σ.

The Matching Pattern Poset
A matching is a pattern of matching, written σ ≤ τ, when σ can be obtained from τ by deleting some of its edges and consistently relabeling the remaining vertices. We denote by M(n, σ) the class of matchings avoiding the pattern σ.

Figure 1: The matching σ = (1 3 2 4) is a pattern of the matching (1 9, 2 5, 3 6, 4 7, 8 10).

Previous results
The following enumerative results are known:
(i) \(|M_n((1212))| = C_n^+\)
(ii) \(|M_n((12132))| = C_n C_{n+2} - C_{n+1}^2\)
(iii) \(|M_n((123123))| = C_n C_{n+2} - C_{n+1}^2\)

Figure 2: The decomposition of a matching λ containing σ and avoiding the juxtaposition of σ and τ, where λA_2 is the pattern of λ containing σ and having minimal leftmost and rightmost vertices.

Proposition. Let σ and τ be matchings and \(n \geq |\sigma|\), then
\[ |M_n(\sigma(\tau + |\sigma|))| = |M_n(\sigma)| + \sum_{i=0}^{n-|\sigma|} \left( \begin{array}{c} 2(n-i) \left( \binom{2i}{i} \binom{2(n-f)-k}{k} \right) k!M_{n-i-k}(\tau) \end{array} \right) \]

Figure 3: The two possible structures for a matching containing minimally containing 1212, where σ and τ are noncrossing matchings.

Figure 4: The matching 12321 is the lifting of the matching 1212.

Lifting of a pattern
The matching 1(σ + |σ|) will be called the lifting of σ. Graphically, its linear chord diagram can be represented by inserting the linear chord diagram of σ into an additional edge. We say that a matching is connected when it is not the juxtaposition of two non-empty matchings.

Theorem 0.1. Let σ ∈ \{1212, 1221\}, τ be a matching and \(n \geq 2\). Then
\[ |M_n(\sigma(\tau + |\sigma|))| = C_n + \frac{1}{n} \sum_{i=0}^{n} \left( \begin{array}{c} 2(n-i) \left( \binom{2i}{i} \binom{2(n-f)-k}{k} \right) k!M_{n-i-k}(\tau) \end{array} \right) \]

Figure 5: The structure of a connected matching avoiding the set of patterns \(\{1(\sigma + |\sigma|); \chi, \gamma\}\) and having root 12323.

Corollary 0.1. Let \(n \geq 2\) and \(\sigma \in \{1212, 1221\}\), then
\[ |M_n(\sigma(\tau + |\sigma|))| = C_n + \frac{1}{n} \sum_{i=0}^{n} \left( \begin{array}{c} 2(n-i) \left( \binom{2i}{i} \binom{2(n-f)-k}{k} \right) k!M_{n-i-k}(\tau) \end{array} \right) \]

Theorem 0.2. The generating function of matchings avoiding the patterns 12321, 12132 and 12231 is given by
\[ M(1(\sigma + |\sigma|); \chi, \gamma, z) = \frac{1}{1 - x C(z) C(z - 1)} \]

Unlabeled pattern avoidance
Let T_n denote the 2n-cycle (1 2 3 ... 2n) on [2n]. We say that σ and τ are cyclically equivalent when there exists k ∈ [2n] such that \(\{i, j\} \in \sigma\) if and only if \(\{i + k, j + k\} \in \tau\), for every \(i, j \in [2n]\). In other words, σ and τ are cyclically equivalent when they have the same unlabeled circular chord diagram. An equivalence class of matchings is called an unlabeled matching.

Proposition. The generating function of matchings avoiding the unlabeled pattern (12323) is given by
\[ M(1(\sigma + |\sigma|); \chi, \gamma, z) = C(z) + \frac{z^2}{(1 - z)^2(1 - 2z)} \]

Figure 7: The structure of a matching avoiding the unlabeled pattern 12323 and containing the pattern 1212, where σ is a permutation matching avoiding the permutations patterns 231 and 312.

Moreover, its coefficients have the following closed form
\[ |M_n((12323))| = C_n + 2^n - n - 1 \]
for \(n \geq 2\).

Figure 8: The bijection ϕ between ternary trees and matchings avoiding the pattern 123123.