Enumerative combinatorics and pattern avoidance in the matching pattern poset

Matteo Cervetti, Luca Ferrari
matteo.cervetti@unitn.it, luca.ferrari@unifi.it

UNIVERSITÀ DEGLI STUDI FIRENZE
DIMAI DIPARTIMENTO DI MATEMATICA E INFORMATICA "ULISSE DINI"

Synopsis

We investigate the combinatorics of the *matching pattern poset*.
Pattern avoidance. Juxtaposition of two pattern: enumeration of two classes. Lifting of a pattern: enumeration of one class. Unlabeled pattern avoidance: enumeration of two classes.

The Matching Pattern Poset

A matching of the set $[2n] = \{1, 2, ..., 2n\}$ is a partition of $[2n]$ into blocks with two elements. A matching σ is a *pattern* of a matching τ, written $\sigma \leq \tau$, when σ can be obtained from τ by deleting some of its edges and consistently relabelling the remaining vertices. We denote by $\mathcal{M}_n(\sigma)$ the class of matchings avoiding the pattern σ.

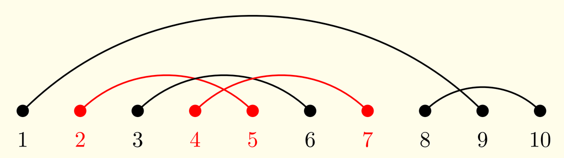

Figure 1: The matching $\{\{1,3\},\{2,4\}\}$ is a pattern of the matching $\{\{1,9\},\{2,5\},\{3,6\},\{4,7\},\{8,10\}\}$.

Previous results

The following enumerative results are known:

(i) $|\mathcal{M}_n(1212)| = C_n$;

(ii) $|\mathcal{M}_n(123123)| = C_n C_{n+2} - C_{n+1}^2$;

(iii) $\mathcal{M}(123132, z) = \frac{54z}{1+36z-(1-12z)^{\frac{3}{2}}}$.

Juxtaposition of two patterns

The matching $\sigma(\tau + |\sigma|)$ will be called the *juxtaposition* of σ and τ. Graphically, its linear chord diagram can be represented by juxtaposing the linear chord diagrams of σ and τ.
We say that τ *minimally contains* σ when σ is a pattern of τ and σ is not a pattern of the matching obtained from τ by deleting its rightmost edge. Denote by $\mu_n(\sigma)$ the set of matchings with order n minimally containing σ.

Proposition. *Let σ and τ be matchings and $n \geq |\sigma|$, then*

$$|\mathcal{M}_n(\sigma(\tau + |\sigma|))| = |\mathcal{M}_n(\sigma)| + \sum_{\ell=|\sigma|}^{n} \sum_{k=0}^{n-\ell} \binom{2\ell+k-1}{k}\binom{2n-2\ell-k}{k} k! |\mu_\ell(\sigma)||\mathcal{M}_{n-\ell-k}(\tau)|.$$

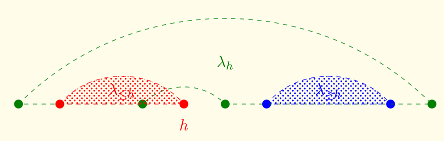

Figure 2: The decomposition of a matching λ containing σ and avoiding the juxtaposition of σ and τ, where $\lambda_{\leq h}$ is the pattern of λ containing σ and having minimal leftmost and rightmost vertices.

Proposition. *Let $n \geq 2$, then $|\mu_n(1212)| = \binom{2n-1}{n-2}$.*

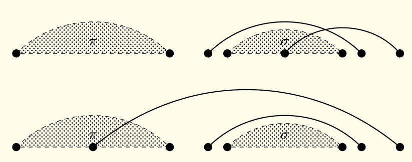

Figure 3: The two possible structures for a matching minimally containing 1212, where σ and π are non-crossing matchings.

Theorem 0.1. *Let $\sigma \in \{1212, 1221\}$, τ be a matching and $n \geq 2$. Then*

$$|\mathcal{M}_n(\sigma(\tau + 2))| = C_n + \sum_{\ell=2}^{n}\sum_{k=0}^{n-\ell}\binom{2\ell-1}{\ell-2}\binom{2\ell-1+k}{k}\binom{2(n-\ell)-k}{k} k! |\mathcal{M}_{n-\ell-k}(\tau)|.$$

Corollary 0.1. *Let $n \geq 2$ and $\sigma \in \{1212, 1221\}$.*

(i) If $\tau \in \{1212, 1221\}$, then $|\mathcal{M}_n(\sigma(\tau+2))| =$

$$C_n + \sum_{\ell=2}^{n}\sum_{k=0}^{n-\ell}\binom{2\ell-1}{\ell-2}\binom{2\ell+k-1}{k}\binom{2n-2\ell-k}{k} k! C_{n-\ell-k}.$$

(ii) If $\tau \in \{123123, 123321\}$, then $|\mathcal{M}_n(\sigma(\tau+2))| =$

$$C_n + \sum_{\ell=2}^{n}\sum_{k=0}^{n-\ell}\binom{2\ell-1}{\ell-2}\binom{2\ell+k-1}{k}\binom{2n-2\ell-k}{k} k! (C_{n-\ell-k}C_{n-\ell-k+2} - C_{n-\ell-k+1}^2).$$

Lifting of a pattern

The matching $1(\sigma+1)1$ will be called the *lifting* of σ. Graphically, its linear chord diagram can be represented by nesting the linear chord diagram of σ into an additional edge. We say that a matching is *connected* when it is not the juxtaposition of two non-empty matchings.

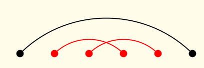

Figure 4: The matching 123231 is the lifting of the matching 1212.

Proposition. *Let σ be a connected matching, $\chi = 123132$ and $\overline{\chi} = 123213$. Then*

$$\mathcal{M}(1(\sigma+1)1, \chi, \overline{\chi}, z) = \frac{1}{1 - z\mathcal{M}(\sigma, \chi, \overline{\chi}, z)C(z\mathcal{M}(\sigma, \chi, \overline{\chi}, z)^2)}.$$

Figure 5: The two additional patterns χ and $\overline{\chi}$.

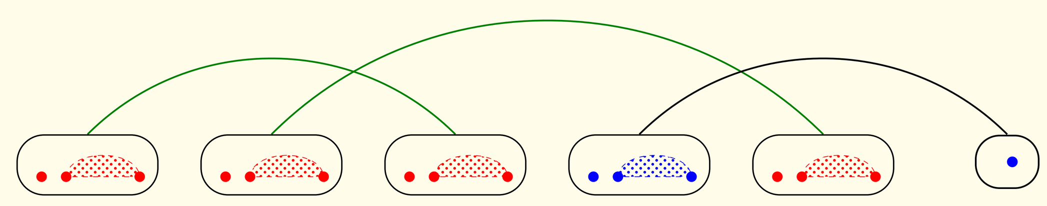

Figure 6: The structure of a connected matching avoiding the set of patterns $\{1(\sigma+1)1, \chi, \overline{\chi}\}$ and having roof 121323.

Theorem 0.2. *The generating function of matchings avoiding the patterns 123231, 123132 and 123213 is given by*

$$\mathcal{M}(123231, 123132, 123213, z) = \frac{1}{1 - zC(z)C(C(z)-1)}$$

and $|\mathcal{M}_n(123231, 123132, 123213)| = |\mathcal{M}_n(123321, 123132, 123213)|$ is the n^{th} term of sequence A125188 in OEIS, counting the Dumont permutations of the first kind avoiding the patterns 2413 and 4132.

Unlabeled pattern avoidance

Let γ_n denote the $2n$−cycle $(1\ 2\ 3\ \ldots\ 2n)$ on $[2n]$.We say that σ and τ are *ciclically equivalent* when there exists $k \in [2n]$ such that $\{i, j\} \in \sigma$ if and only if $\{\gamma_n^k(i), \gamma_n^k(j)\} \in \tau$, for every $i, j \in [2n]$. In other words σ and τ are ciclically equivalent when they have the same unlabeled circular chord diagram. An equivalence class of matchings is called an *unlabeled matching*.

Proposition. *The generating function of matchings avoiding the unlabeled pattern [112323] is given by*

$$\mathcal{M}([112323], z) = C(z) + \frac{z^2}{(1-z)^2(1-2z)}$$

Moreover, its coefficients have the following closed form

$$|\mathcal{M}_n([112323])| = C_n + 2^n - n - 1$$

for $n \geq 2$.

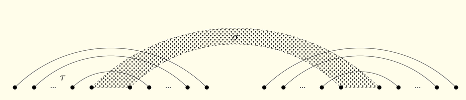

Figure 7: The structure of a matching avoiding the unlabeled pattern [112323] and containing the pattern 1212, where σ is a permutational matching avoiding the permutation patterns 231 and 312.

Proposition. *For every $n \geq 1$ there is an explicit bijection between matchings of order n avoiding the unlabeled pattern [123132] and the class of ternary trees with n nodes, in particular*

$$|\mathcal{M}_n([123132])| = \frac{1}{2n+1}\binom{3n}{n}.$$

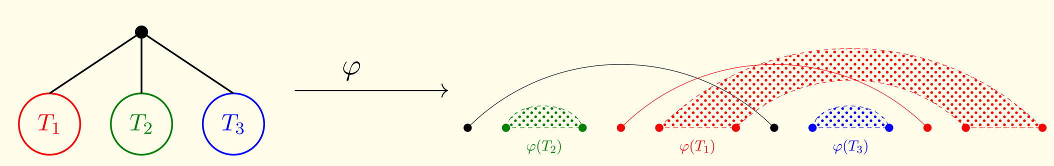

Figure 8: The bijection φ between ternary trees and matchings avoiding the pattern [123132].