K block set partition patterns and statistics

Amrita Acharyya, Robinson Paul Czajkowski, Allen Richard Williams
Department of Mathematics, University of Toledo

Introduction

In [1] pattern avoidance in set partitions combined with four important statistics defined by Wachs and White in [4] has been studied. Sagan, Dahlberg, Dorward, Gerhard, Grubb, Purcell, Reppuhn consider the distributions of these statistics over various avoidance classes. The analogues of their many results in [1] follow for set partitions with exactly k blocks for a specified positive integer k, towards Wach's and White's statistics and a minor variation of their Statistics. We discuss here some of these analogues.

Definition

A set partition of a set S is a collection σ of nonempty subsets whose disjoint union is S. A set partition of S can be written as $\sigma = B_1/B_2/...B_k \vdash S$ where the subsets B_i are called blocks.For $[n]=\{1,...,n\}$, the notation $\Pi_n=\{\sigma:\sigma\vdash[n]\}$ means the set of all set partitions of $[n]$.

Definition

A restricted growth function (RGF) is a sequence $w=a_1...a_n$ of positive integers subject to the restrictions

1. $a_1=1$.
2. For $i\geqslant 2$, $a_i\leqslant 1+\max\{a_1,...,a_{i-1}\}$

Remarks

We let

i $\Pi_n(\pi)=\{\sigma\in\Pi_n : \sigma \text{ avoids } \pi\}$.

ii $\Pi_{n,k}(\pi)$ be the set of all words in Π_n with exactly k blocks.

iii $R_n=\{w : w \text{ is an RGF of length } n\}$.

iv $R_{n,k}=\{w : w \text{ is an RGF of length } n \text{ with maximal letter } k\}$.[Note:As there is a bijection between Π_n and R_n, the restriction of that bijection defines a one to one correspondence between $\Pi_{n,k}$ and $R_{n,k}$ naturally.]

v The four statistics of Wachs and White are denoted as lb, ls, rb and rs. Given a word $w=a_1...a_n$, we define $lb(a_j)=\#\{a_i : i<j$ and $a_i>a_j\}$ and the rest accordingly. As a minor variation of these Statistics we consider $lbe(a_j)=\#\{a_i : i<j \text{ and } a_i\geqslant a_j\}$ and the rest accordingly. Also,we define $LB_{n,k}(\pi)=LB_{n,k}(\pi,q)=\sum_{\sigma\in\Pi_{n,k}(\pi)} q^{lb(\sigma)}$ and the three analogous polynomials for the other three Statistics as in [1]. We define accordingly, all four respective polynomials for the minor variation of Wach's and White's Statistics.

vi Often, we will even be able to compute the multivariate generating function $F_{n,k}(\pi)=F_{n,k}(\pi,q,r,s,t)=\sum_{\sigma\in\Pi_{n,k}(\pi)} q^{lb(\sigma)}r^{ls(\sigma)}s^{rb(\sigma)}t^{rs(\sigma)}$ and analogous $FE_{n,k}(\pi)=FE_{n,k}(\pi,q,r,s,t)=\sum_{\sigma\in\Pi_{n,k}(\pi)} q^{lbe(\sigma)}r^{lse(\sigma)}s^{rbe(\sigma)}t^{rse(\sigma)}$

vii The generating function of $FE_{n,k}$ can be found by adding $n-k$ in the exponent of each variable, as The generating function of FE_n can be found from F_n by adding $n-k$ in the exponent of each of the four variables, where k is the largest letter in the corresponding RGF. And accordingly for the other one variable generating functions as well.

The pattern 1/23

i. $F_{n,k}(1/23)=(rs)^{\binom{k}{2}}$, when n=k, otherwise when $n>k$,this is $\sum_{j=1}^{k}(qt)^{j-1}r^{\binom{k}{2}}s^{(n-k)(k-1)+(k-j)+\binom{k-1}{2}}$

ii. $LB_{n,k}(1/23)=RS_{n,k}(1/23)$, when $n=k$ they are 1, otherwise they are $\sum_{j=1}^{k-1}q^{j-1}$.

iii. $LS_{n,k}(1/23)=r^{\binom{k}{2}}$, when $n=k$, otherwise this is $kr^{\binom{k}{2}}$

iv. $RB_{n,k}(1/23)=(s)^{\binom{k}{2}}$, when $n=k$,otherwise this is $\sum_{j=1}^{k}s^{(n-k)(k-1)+(k-j)+\binom{k-1}{2}}$

The pattern 12/3

i. $F_{n,k}(12/3)=(rs)^{\binom{k}{2}}$, when n=k, otherwise, this is $\sum_{i=1}^{k}q^{(n-k)(k-i)}r^{\binom{k}{2}+(n-k)(i-1)}s^{\binom{k}{2}}t^{k-i}$.

ii. $LS_{n,k}(12/3)=r^{\binom{k}{2}}$, when n=k, otherwise this is $\sum_{i=1}^{k}r^{\binom{k}{2}+(n-k)(i-1)}$

The pattern 123

i $LS_{n,k}(123)=\sum_L(\Pi_{g=1}^{n-k})(k-l_g+g)q^{\binom{k}{2}+\sum_{l\in L}(l-1)}$, where the sum is over all subsets $L=\{l_1,l_2,...l_{n-k}\}$ of [k] with $l_1>l_2>...>l_{n-k}$

ii For $k\geqslant\lceil\frac{n}{2}\rceil$, the degree of $LB_{n,k}(123)=\frac{(4n+1)k-3k^2-n^2-n}{2}$.

iii The leading coefficient of $LB_{n,k}(123)$ is $(n-k)!$

iv For $k\geqslant\lceil\frac{n}{2}\rceil$, the degree of $RS_{n,k}(123)=(n-k)(k-1)$.

7. Avoidance classes avoiding two partitions of [3] and associated RGFs of exactly k blocks.

Avoidance class	Assiciated RGF's
$\Pi_{n,k}(1/2/3,1/23)$	1^n (k=1) , $1^{n-1}2, 1^{n-2}21$ (k=2)
$\Pi_{n,k}(1/2/3,13/2)$	1^n (k=1), $1^m2^{n-m}, 1<m\leqslant n$ (k=2)
$\Pi_{n,k}(1/2/3,12/3)$	1^n (k=1), $12^{n-1}, 121^{n-2}$ (k=2)
$\Pi_{n,k}(1/23,13/2)$	$1^{n-k+1}23...k$.
$\Pi_{n,k}(1/23,12/3)$	1^n(k=1), 123...$(n-1)$1 ($k=$ n-1), 123...n (k=n)
$\Pi_{n,k}(13/2,12/3)$	1^n, (k=1), 123...k^{n-k+1}
$\Pi_{n,k}(13/2,123)$	Layered RGF's, each layer with atmost 2 elements, $k\geqslant\lceil\frac{n}{2}\rceil$
$\Pi_{n,k}(12/3,123)$	123...k (k=n), 123...$ki, 1\leqslant i\leqslant k$ (k=n-1)

We have analogue to some part of Theorem 7.1 in [1]:
For $n\geqslant 3$

1. $F_{n,k}(1/2/3,12/3)=1,(k=1)$ and it is $rs^{n-1}+qrs^{n-2}t$ $(k=2)$.
2. $F_{n,k}(1/23,12/3)=1$ for k =1
   $=(rs)^{\binom{k}{2}}$ fo $k=n$
   $=(qt)^{k-1}(rs)^{\binom{k}{2}}$ for k=n-1
3. $F_{n,k}(13/2,123)=\sum_L r^{\binom{k}{2}+\sum_{l\in L}(l-1)}s^{\binom{k}{2}+\sum_{l\in L}(m-l)}$, where L is over all subsets $L=\{l_1,l_2,...l_{n-k}\}$ of [k] with $l_1>l_2>...>l_{n-k}$.

Reference

1. Samantha Dahlberg, Robert Dorward, Jonathan Gerhard, Thomas Grubb, Carlin Purcell, Lindsey Reppuhn, Bruce E.Sagan. *Set partition patterns and statistics.* 2015.

2. Samantha Dahlberg, Robert Dorward, Jonathan Gerhard, Thomas Grubb, Carlin Purcell, Lindsey Reppuhn, Bruce E.Sagan. *Restricted growth function patterns and statistics* , 2016.

3. Bruce E. Sagan, *Pattern avoidance in set partitions* Ars Combin.94(2010)7996.

4. Michelle Wachs, Dennis White *p,q-Stirling numbers and set partition statistics.*J.Combin.Theory Ser.A56(1)(1991)2746.

Email: amrita.acharyya@utoledo.edu, robinson.czajkowski@rockets.utoledo.edu, allen.williams2@rockets.utoledo.edu