P- and Q-vexillary involutions

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Reduced words and involution words

- $s_i := (i, i+1)$ a transposition in the symmetric group
- w and z will denote permutations, with z being an involution
- $\mathcal{R}(w)$ the set of reduced words $a = a_1 \cdots a_\ell$ of w: minimal-length with $s_{a_1} \cdots s_{a_\ell} = w$.
- $\widehat{\mathcal{R}}(z)$ the set of involution words $a = a_1 \cdots a_\ell$ of z: minimal-length with $s_{a_\ell} \circ \cdots \circ s_{a_1} \circ s_{a_1} \circ \cdots \circ s_{a_\ell} = z$ where $\circ$ is the associative product for which $w \circ s_i$ means whichever of w and ws_i has more inversions.

Example

212 $\in \mathcal{R}(321)$, while **12** $\in \widehat{\mathcal{R}}(321)$ since $s_2 \circ s_1 \circ s_1 \circ s_2 = s_2 s_1 s_2$ is a reduced expression for z.

Stanley symmetric functions

Definition

The Stanley symmetric function F_w and involution Stanley symmetric function $\widehat{F}_z$ are given by

$$\sum_{a} L_{\mathrm{Asc}(a)} = \sum_{a} \sum_{\substack{1 \le i_1 \le \cdots \le i_\ell \\ a_j < a_{j+1} \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_\ell},$$

where a ranges over $\mathcal{R}(w)$ (resp. $\widehat{\mathcal{R}(z)}$), L_S is a fundamental quasisymmetric function, and $\mathrm{Asc}(a)$ is the ascent set of a.

Example

$\mathcal{R}(2143) = \{\mathbf{13}, \mathbf{31}\}$ and $\widehat{\mathcal{R}}(321) = \{\mathbf{12}, \mathbf{21}\}$, so

$$F_{2143} = \widehat{F}_{321} = \sum_{i_1 < i_2} x_{i_1} x_{i_2} + \sum_{i_1 \le i_2} x_{i_1} x_{i_2} = e_2 + h_2 = s_{11} + s_2.$$

Stanley symmetric function

Ordinary Stanley symmetric functions:

- F_w is Schur positive (Edelman-Greene) and every skew Schur function arises as some F_w
- F_w represents the cohomology class of a graph Schubert variety in a Grassmannian (Knutson-Lam-Speyer)
- F_w equals a single Schur function if and only if w is vexillary (2143-avoiding) (Stanley)

Involution Stanley symmetric functions:

- $\widehat{F}_z$ is positive in the basis of Schur P-functions (Hamaker-Marberg-Pawlowski)
- $2^{\#\text{ of 2-cycles in } z}\widehat{F}_z$ represents the cohomology class of a symmetric graph Schubert variety in a Lagrangian Grassmannian (Pawlowski)
- $\widehat{F}_z$ equals a single Schur P-function if and only if...?

P- and Q-vexillary involutions

- Say an involution $z \in S_n$ is P-vexillary if $\widehat{F}_z$ equals a single Schur P-function P_λ.
- Geometry suggests considering $2^{\#\text{ of 2-cycles in } z}\widehat{F}_z$; say z is Q-vexillary if this is a single Schur Q-function $Q_\lambda := 2^{\ell(\lambda)} P_\lambda$.

Example

$(1,3)$ is P- and Q-vexillary:

$$\widehat{F}_{(1,3)} = P_2, \quad 2^1 \widehat{F}_{(1,3)} = 2P_2 = Q_2$$

$(1,2)(3,4)$ is P-vexillary but not Q-vexillary:

$$\widehat{F}_{(1,2)(3,4)} = P_2, \quad 2^2 \widehat{F}_{(1,2)(3,4)} = 4P_2 = 2Q_2.$$

P- and Q-vexillary involutions

Theorem (Hamaker-Marberg-Pawlowski)

z is Q-vexillary or P-vexillary if and only if its restriction to every z-invariant set of positions of size at most 8 has the same property.

The constraint that bad patterns appear in a z-invariant set of positions makes the theorem easier to check: is this an interesting variant on pattern avoidance?

Theorem (Hamaker-Marberg-Pawlowski)

- *z is Q-vexillary if and only if it is vexillary (2143-avoiding).*
- *z is P-vexillary if and only if it does not contain any of the following patterns in a z-invariant set of positions:*

(1,2)(3,5)	(1,4)(2,3)(5,6)	(1,5)(2,4)(3,7)	(1,6)(2,5)(3,8)(4,7)
(1,3)(4,5)	(1,2)(3,6)(4,5)	(1,5)(3,7)(4,6)	(1,6)(2,4)(3,8)(5,7)
(1,4)(3,6)	(1,2)(3,4)(5,6)		(1,3)(2,5)(4,7)(6,8)

An application to Schur P-positivity of skew Schur functions

- One can show (e.g. Ardila–Serrano, or by using facts about $\widehat{F}_z$) that if δ_n is a staircase partition $(n-1, n-2, \ldots, 1)$ and $\mu \subseteq \delta_n$, then the skew Schur function $s_{\delta_n/\mu}$ is Schur P positive.
- As an application of our results, we reprove a theorem of DeWitt: $s_{\delta/\mu}$ equals a single Schur P function if and only if δ_n/μ equals some δ_m/rectangle up to permuting rows and columns.