Atoms for signed permutations

Zachary Hamaker
University of Florida

Eric Marberg
Hong Kong University of Science and Technology

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Atoms for involutions in Coxeter groups

Let (W, S) be a Coxeter group with length function $\ell : W \to \mathbb{Z}_{\geq 0}$.
There is a unique associative product $\circ : W \times W \to W$ with

$$\begin{cases} v \circ w = vw \text{ if } v, w \in W \text{ and } \ell(vw) = \ell(v) + \ell(w) \\ s \circ s = s \text{ for } s \in S. \end{cases}$$

An *atom* for $z \in W$ is an element w of minimal length with $w^{-1} \circ w = z$.
Let $\mathcal{A}(z)$ denote the set of atoms for z. Then $\mathcal{A}(z) \neq \varnothing$ iff $z = z^{-1}$.

A *reduced word* for w is a minimal length expr. $w = s_1 s_2 \cdots s_l$ with $s_i \in S$
An element $w \in W$ is *fully commutative* if any two of its reduced words are connected by a sequence of moves swapping adjacent commuting factors.

Theorem. If $|\mathcal{A}(z)| = 1$ then $z = z^{-1} \in W$ is fully commutative.

Atomic involutions in the symmetric group

Suppose $W = S_n$ is the symmetric group of permutations of $\{1, 2, \ldots, n\}$. This is a Coxeter group for $S = \{s_1, s_2, \ldots, s_{n-1}\}$ where $s_i = (i, i+1)$.

Theorem. If $z = z^{-1} \in S_n$ then the following are equivalent:

(a) $|\mathcal{A}(z)| = 1$.

(b) z is fully commutative.

(c) z is 321-avoiding.

(d) z has no nesting cycles, i.e., one never has $a < b \leq z(b) < z(a)$.

Say that an involution $z = z^{-1} \in W$ is *atomic* if $|\mathcal{A}(z)| = 1$.

Corollary. The number of atomic involutions in S_n is $\binom{n}{\lfloor n/2 \rfloor}$.

Atomic involutions in the group of signed permutations

A *signed permutation* is a bijection $w : \mathbb{Z} \to \mathbb{Z}$ with $w(-i) = -w(i)\ \forall i$.

Let W_n be the group of signed permutations w with $w(i) = i$ for $|i| > n$.

Coxeter group for $S = \{(-1,1)\} \sqcup \{(-i,-i-1)(i,i+1) : 0 < i < n\}$.

Theorem (Stembridge). If $z = z^{-1} \in W_n$ then TFAE:

(a) z is fully commutative.

(b) One never has $a < b \leq z(b) < z(a)$.

The number of such involutions is $2^n + \binom{n}{\lfloor n/2 \rfloor} - 1$.

Theorem (H.-M.). If $z = z^{-1} \in W_n$ then TFAE:

(a) $|\mathcal{A}(z)| = 1$.

(b) One has $a < b \leq z(b) < z(a)$ only if $z(a) = -a$ and $z(b) \neq -b$.

The number of such involutions is $\begin{cases} (n+3)2^{n-2} & \text{when } n \text{ is odd} \\ (n+4)2^{n-2} - \frac{n}{4}\binom{n}{n/2} & \text{when } n \text{ is even.} \end{cases}$

Orbits closures in flag varieties

Atoms in finite Coxeter groups appear in certain cohomology formulas.

Let $G = \mathrm{GL}_n(\mathbb{C})$. Let $B \subseteq G$ be the subgroup of upper triangular matrices.

The opposite Borel subgroup B^- of lower triangular matrices acts on the flag variety G/B with finitely many orbits $\mathring{X}_w$, uniquely indexed by $w \in S_n$.

The orthogonal group $K = \mathrm{O}_n(\mathbb{C})$ also acts on G/B with finitely many orbits $\mathring{Y}_z$, uniquely indexed by involutions $z = z^{-1} \in S_n$.

Let X_w and Y_z be the closures of these orbits. Let $\kappa(z) = |\{a : a < z(a)\}|$.

Theorem (Brion). Then $[Y_z] = \sum_{w \in \mathcal{A}(z)} 2^{\kappa(z)} [X_w] \in H^*(G/B)$.

Atoms for signed permutations and orbit closures

Let $G = \mathrm{Sp}_{2n}(\mathbb{C})$. Choose a Borel subgroup $B \subseteq G$. The opposite Borel subgroup B^- acts on G/B with finitely many orbits $\mathring{X}_w$, uniquely indexed by $w \in W_n$.

There is a subgroup $K = \mathrm{GL}_n(\mathbb{C}) \subset G$ acting on G/B with finite set of orbits $\mathring{Y}_\gamma$, uniquely indexed by pairs $\gamma = (z, \varepsilon)$, where $z = z^{-1} \in W_n$ and ε is a map assigning ± 1 to each $0 < a \leq n$ with $z(a) = -a$. Write X_w and Y_γ for closures.

Theorem (Brion). One has $[Y_\gamma] = \sum_{w \in \mathcal{A}(\gamma)} 2^{\kappa(z) - \ell_0(w)}[X_w] \in H^*(G/B)$.

Here $\ell_0(w) = |\{a > 0 : w(a) < 0\}|$ and $\mathcal{A}(\gamma) \subseteq \mathcal{A}(z)$ for $\gamma = (z, \varepsilon)$.

To define $\mathcal{A}(\gamma)$, fix $w \in \mathcal{A}(z)$. Write $a_1 a_2 \cdots a_n = w^{-1}(1) w^{-1}(2) \cdots w^{-1}(n)$.
Inductively remove consecutive pairs $a_i > a_{i+1}$ until the sequence is increasing.
For example, $a_1 a_2 a_3 a_4 a_5 = -1, 2, 5, -4, -3, 6 \rightsquigarrow -1, 2, -3, 6 \rightsquigarrow -1, 6$.
Define $\mathrm{sh}(w)$ to be set of pairs (a, b) removed with $0 < a_i < -a_{i+1}$.

Theorem (H.-M.) If $(a, b) \in \mathrm{sh}(w)$ then $z(a) = -a$ and $z(b) = -b$, and

$$\mathcal{A}(\gamma) = \{w \in \mathcal{A}(z) : (a, b) \in \mathrm{sh}(w) \ \Rightarrow \ \varepsilon(a) \neq \varepsilon(-b)\}.$$

References

Some related references:

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