Boolean elements in the Bruhat order

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Boolean Weyl group elements

For a Weyl group $W$, an element $w \in W$ is called boolean if the interval $[id,w]$ in the (strong) Bruhat order is isomorphic to a Boolean lattice. The following characterization of boolean elements is a simple consequence of the subword property of the strong Bruhat order.

Proposition

An element $w \in W$ is boolean if and only if any reduced expression (or equivalently, all reduced expressions) of $w$ does not contain repeated letters.

For type $A_n$, $G = \mathfrak{S}_{n+1}$. Tenner [1] proved that a permutation $w \in A_n$ is boolean if it avoids 321 and 3412, and gave a signed pattern avoidance characterization of boolean elements in types $B$ and $D$, with the number of patterns being 10 and 20, respectively. By analyzing inversion sets, we obtain a more succinct pattern avoidance characterization of boolean elements. In our most concise version, boolean elements are characterized by avoiding just 3 linear patterns:

- $321, s_2s_1w \in W(A_3), 3412 = s_3s_2s_1w \in W(A_4)$,
- and the new $s_2s_1s_3w \in W(D_2)$.

This will be made precise after discussing some background.

Background on the Bruhat order and inversion sets

The (strong) Bruhat order on a Weyl group $W$ is the transitive closure of all $w \not\leq t$, $\ell(t(w)) = \ell(w) + 1$ where $t$ is some reflection and $\ell$ is the Coxeter length. The identity Weyl group element $e$ is the minimum of this partial order.

We let $\Phi$ be a root system, with positive roots $\Phi^+ \subset \Phi$. Let $W = W(\Phi)$ be its Weyl group. For $w \in W$, the inversion set is $I(w) = \{ \beta \in \Phi^+ \mid w^{-1}\beta < 0 \}$. The next proposition is useful and well-known (see for example [2]).

Linear patterns can be seen as a relaxation of BP patterns.

Definition of BP pattern containment

We say that $w \in W(\Phi)$ contains the BP (Billey-Postnikov) pattern $\pi \in W(\Phi)$ if there exists a subspace $E \leq \Phi^+$ such that there is an isomorphism between root systems $R$ and $\Phi'$ that preserves positive roots and maps $\pi$ to $w_\pi(E)$. We introduce a new notion of linear patterns, which will enable a very efficient characterization of boolean elements.

Definition of linear pattern containment

We say that $w \in W(\Phi)$ contains the linear pattern $\pi \in W(R)$ if there exists a linear transformation $R \rightarrow \Phi'$ that maps positive roots to positive roots, inversions $I_\pi^R(x) \in \pi$ to inversions $I_\Phi^\prime(\pi)$, and non-inversions $\Phi' \setminus I_\Phi(\pi)$.

A linear pattern avoidance characterization of boolean elements

The first version of our main theorem is a characterization of boolean elements in terms of linear pattern avoidance.

A BP pattern avoidance characterization of boolean elements

From the linear pattern version, one can deduce a BP pattern characterization of boolean elements, which is the second version of our main theorem.

Characterization theorem

Linear pattern version

Let $\Phi$ be an irreducible root system. An element $w \in W(\Phi)$ is boolean if and only if it avoids the linear patterns $s_2s_1w \in W(A_3), s_2s_1s_3w \in W(A_4)$, and $s_2s_1s_3s_2w \in W(D_2)$.

A BP pattern version

An element $w \in W$ is boolean if and only if $w$ avoids the BP patterns in Table 1.

<table>
<thead>
<tr>
<th>type $\Pi$</th>
<th>forbidden patterns</th>
<th># patterns</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$s_3s_2s_1s_3s_2s_1 = 321$</td>
<td>1</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$s_1s_2s_1s_2s_3(3412) \ ; s_1s_2s_3s_2s_1 = s_2s_3s_1s_2w$</td>
<td>5</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$s_1s_2s_3w$ \ ; $s_1s_2s_3s_1s_2s_3 = s_2s_3s_1s_2w$</td>
<td>3</td>
</tr>
<tr>
<td>$C_2$</td>
<td>all patterns of Coxeter length at least 3</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 1: Forbidden patterns for boolean elements in Weyl groups

$\Pi$-boolean elements

We say a permutation $w \in \mathfrak{S}_n$ is $\Pi$-boolean if for any reduced word $w$ of $w$, there is no simple transposition $s_i$ that appears strictly more than $k$ times. This generalizes boolean elements, as $w$ is boolean if and only if it is 1-boolean. It turns out that 2-boolean elements of $\mathfrak{S}_n$ can also be characterized in terms of pattern avoidance.

Pattern avoidance characterization of 2-boolean elements

A permutation $w \in \mathfrak{S}_n$ is 2-boolean if and only if $w$ avoids 3421, 4312, 4321 and 456123.

Enumeration of 2-boolean elements

Let $f(n)$ be the number of 2-boolean permutations in $\mathfrak{S}_n$.

$$f(n) = \frac{1 - 5q + 5q^2 - 3q^3}{1 - 6q + 9q^2 - 3q^3}$$

Open problems

• Is there a pattern avoidance characterization of 2-boolean elements in any Weyl group?
• Is there a way to enumerate 2-boolean elements without going through pattern avoidance?
• Can one enumerate $k$-boolean elements for any $k > 2$?
• Are there examples showing that $k$-boolean elements are not characterized by pattern avoidance for any $k \geq 4$?

References


Acknowledgements

We would like to thank Alex Postnikov for ideas and suggestions.

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 advocates for tolerance and inclusion.

Enumerating 2-boolean elements