Self-dual intervals in the Bruhat order

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Smooth permutations
A permutation $w \in \mathfrak{S}_n$ is smooth if the corresponding Schubert variety $X_w$ is smooth. The following characterization of smooth permutations is well-known.

**Theorem ([1, 2])**
The following are equivalent for $w \in \mathfrak{S}_n$:

1. The interval $[e, w]$ in the Bruhat order is rank-symmetric.
2. $w$ avoids 3412 and 4231.
3. $w$ is smooth.

Background and Notations on the Bruhat order
The (strong) Bruhat order on the symmetric group $\mathfrak{S}_n$ is the transitive closure of $w \leq w't$, $\ell(w) = \ell(w't) + 1$ where $t$ is the transposition $(i \ j)$ and $\ell$ denotes the number of inversions. There is a minimum $e = 12 \cdots n$ and a maximum $w_0 = n \cdots 1$ in the Bruhat order.

For $w \in \mathfrak{S}_n$, let $P_w$ be a bipartite graph on $P_w^p \cup P_w^r$ and $P_{w_0}^p \cup P_{w_0}^r$ respectively with edges given by cover relations of the Bruhat order.

Let $\deg(u)$ be the number of $v \in [e, w]$ covering $u$, and $\deg_0(u)$ be the number of $v \in [e, w]$ covered by $u$.

Principal order ideals in the Bruhat order are known to be "top-heavy", described by the following theorem.

**Theorem [3]**
For $w \in \mathfrak{S}_n$ and $0 \leq k \leq \ell(w)/2$, $|P_w^p| \leq |P_{w_0}^p| - k$.

Self-dual intervals
A poset is self-dual if it admits an order-reversing bijection. Our main theorem is the following.

**Theorem**
The following are equivalent for $w \in \mathfrak{S}_n$:

1. The bipartite graphs $P_w$ and $P_w^r$ are isomorphic.
2. $w$ avoids the smooth patterns 3412 and 4231 as well as 34521, 45321, 54123, and 54321;
3. $w$ is polished (defined momentarily);
4. The interval $[e, w]$ in the Bruhat order is self-dual.

The equivalence of (1) and (4) is notable because self-duality of $[e, w]$ may be demonstrated by comparing only two pairs of ranks and coranks. For smoothness, Billey and Postnikov [4] conjecture that one must check that $|P_w^p| = |P_{w_0}^p|$ for around the first $n$ pairs of ranks and coranks.

**Theorem**
Let $w \in \mathfrak{S}_n$ be smooth, then

$$\max_{u \in P_w^p} \deg_0(u) \leq \max_{u \in P_{w_0}^p} \deg(u),$$

with equality if and only if $[e, w]$ is self-dual.

Polished elements
Let $(W, S)$ be a finite Coxeter system where $W$ is the Coxeter group and $S$ is the set of simple generators.

**Definition**
We say that $w \in W$ is polished if there exist pairwise disjoint subsets $S_1, \ldots, S_k \subset S$ such that each $S_i$ is a connected subset of the Dynkin diagram and coverings $S_i = J_i \cup J_f'$ for $i = 1, \ldots, k$ with $J_i \cap J_f'$ totally disconnected so that $w = \prod_{i=1}^k u_i w_i(j_i \cap J_f') w_i(j_f')$ where the product is taken from left to right as $i = 1, 2, \ldots, k$.

Note that if the $J_i$'s are reordered, we obtain a possibly different polished element.

The following open questions are natural to ask:

- If $[e, w]$ is self-dual, is $w$ polished?
- Are the conditions in (1) and (4) equivalent?
- Is there a geometric interpretation for self-duality (even in type $A_2$)?

Discussion and open problems
We showed that the implication $(3) \Rightarrow (4)$ ($w$ being polished implies that $[e, w]$ is self-dual) holds for any finite Coxeter group, while $(1) \Rightarrow (4)$ does not hold in general.

We are grateful to Sara Billey for suggesting that self-dual intervals may be characterized by pattern avoidance. We also wish to thank Alexander Woo for providing helpful references and Alexander Postnikov and Thomas Lam for their suggestions.

References


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Figure 1: The bipartite graph $P_{23145}$ (top) and $P_{24132}$ (bottom). Note that $\max_{u \in P_{23145}} \deg_0(u) = 5$ and $\max_{u \in P_{24132}} \deg_0(u) = 6$ in $P_{23145}$ and $P_{24132}$ are not isomorphic.

Figure 2: A polished element in type $A_4$. The duality map
Notice that if $[e, w] = [e, w'] \times [e, w'']$ if $w = w'w''$ for $w'$ and $w''$ living in disjoint parabolic subgroups. So to produce the duality map, it suffices to assume that $w = \prod_{i=1}^k u_i w_i(j_i \cap J_f') w_i(j_f')$ such that $J_i \cap J_f'$ is totally disconnected (any two simple generators commute).

This map is given by $u \mapsto u':= w_1 W_1 w_2 (J_f \cap J') w_1 w_2 (J_f')$ where $w = u' w_1$. u is the parabolic decomposition.