Self-dual intervals in the Bruhat order

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Smooth permutations

A permutation $w \in \mathfrak{S}_n$ is **smooth** if the corresponding Schubert variety X_w is smooth. The following characterization of smooth permutations is well-known.

Theorem ([1, 2])

The following are equivalent for $w \in \mathfrak{S}_n$:

1. the interval $[e, w]$ in the Bruhat order is rank-symmetric
2. w avoids 3412 and 4231;
3. w is smooth.

Background and Notations on the Bruhat order

The **(strong) Bruhat order** on the symmetric group $\mathfrak{S}_n$ is the transitive closure of

$$w \lessdot wt_{ij}, \quad \ell(w) = \ell(wt_{ij}) - 1$$

where t_{ij} is the transposition $(i\ j)$ and ℓ denotes the number of inversions. There is a minimum $e = 12 \cdots n$ and a maximum $w_0 = n \cdots 21$ in the Bruhat order.
For $w \in \mathfrak{S}_n$, and $k = 0, 1, \ldots, \ell(w)$, let

$$P_k^w := \{u \leq w \mid \ell(u) = k\}$$

be the k^{th} rank of w. Then $P_0^w = \{e\}$ and $P_{\ell(w)}^w = \{w\}$.

Let Γ_w and Γ_w be the bipartite graphs on $P_1^w \sqcup P_2^w$ and $P_{\ell(w)-1}^w \sqcup P_{\ell(w)-2}^w$ respectively with edges given by cover relations of the Bruhat order.

Let $\mathrm{udeg}_w(u)$ be the number of $v \in [e, w]$ covering u, and $\mathrm{ddeg}_w(u)$ be the number of $v \in [e, w]$ covered by u.

Principal order ideals in the Bruhat orders are known to be "top-heavy", described by the following theorem.

Theorem [3]

For $w \in \mathfrak{S}_n$ and $0 \leq k \leq \ell(w)/2$,

$$|P_k^w| \leq |P_{\ell(w)-k}^w|.$$

Self-dual intervals

A poset is **self-dual** if it admits an order-reversing bijection. Our main theorem is the following.

Theorem

The following are equivalent for $w \in \mathfrak{S}_n$:

1. the bipartite graphs Γ_w and Γ^w are isomorphic;
2. w avoids the smooth patterns 3412 and 4231 as well as 34521, 45321, 54123, and 54312;
3. w is **polished** (defined momentarily);
4. the interval $[e, w]$ in Bruhat order is self-dual.

The equivalence of (1) and (4) is notable because self-duality of $[e, w]$ may demonstrated by comparing only two pairs of ranks and coranks. For smoothness, Billey and Postnikov [4] conjecture that one must check that $|P_i^w| = |P_{\ell(w)-i}^w|$ for around the first n pairs of ranks and coranks.

Theorem

Let $w \in \mathfrak{S}_n$ be smooth, then

$$\max_{u \in P_1^w} \mathrm{udeg}_w(u) \leq \max_{u \in P_{\ell(w)-1}^w} \mathrm{ddeg}_w(u),$$

with equality if and only if $[e, w]$ is self-dual.

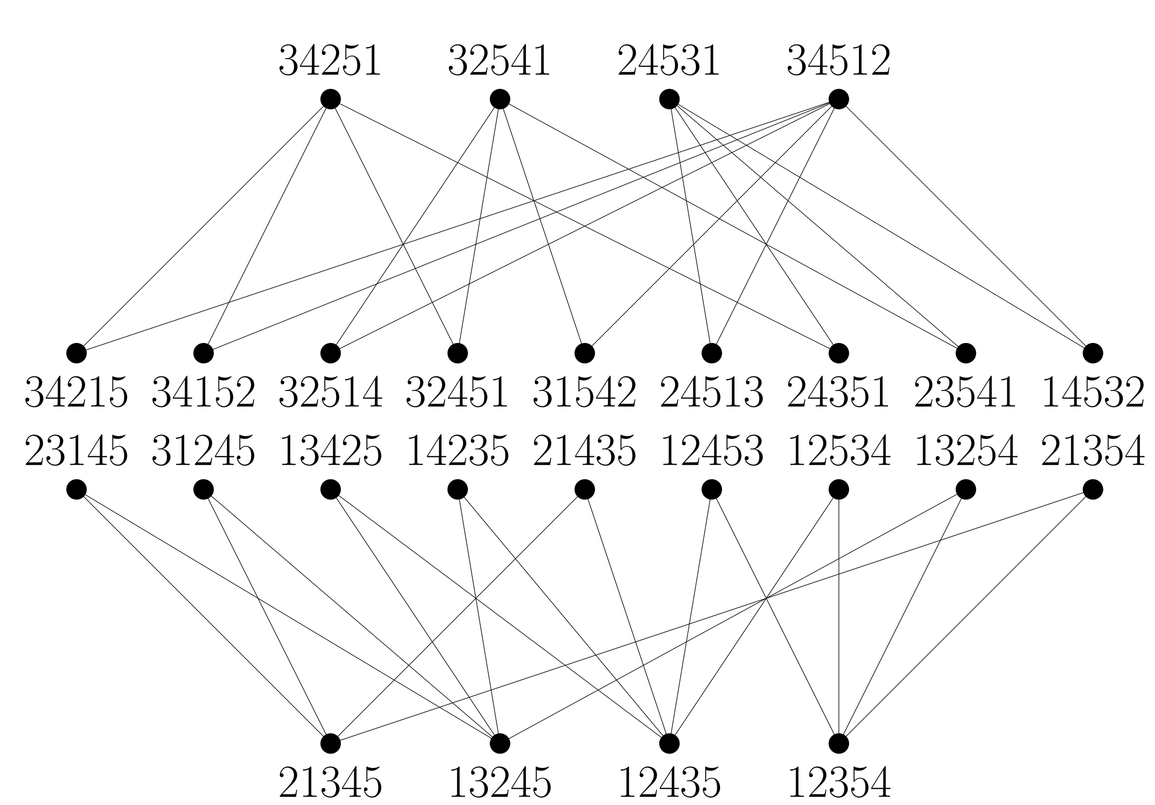

Figure 1: The bipartite graphs Γ^{34521} (top) and Γ_{34521} (bottom). Note that $\max_{u \in P_1^w} \mathrm{udeg}_w(u) = 5$ and $\max_{u \in P_{\ell(w)-1}^w} \mathrm{ddeg}_w(u) = 6$ so Γ^w and Γ_w are not isomorphic.

Polished elements

Let (W, S) be a finite Coxeter system where W is the Coxeter group and S is the set of simple generators.

Definition

We say that $w \in W$ is **polished** if there exist pairwise disjoint subsets $S_1, \ldots, S_k \subseteq S$ such that each S_i is a connected subset of the Dynkin diagram and coverings $S_i = J_i \cup J_i'$ for $i = 1, \ldots, k$ with $J_i \cap J_i'$ totally disconnected so that

$$w = \prod_{i=1}^{k} w_0(J_i) w_0(J_i \cap J_i') w_0(J_i')$$

where the product is taken from left to right as $i = 1, 2, \ldots, k$.

Note that if the S_j's are reordered, we obtain a possibly different polished element.

The following element with $k = 2$, $J_1 = \{s_7, s_8\}$, $J_1' = \emptyset$, $J_2 = \{s_2, s_3, s_4\}$, $J_2' = \{s_4, s_5, s_6\}$, and multiplication in the order of

$$\begin{aligned} w =& w_0(J_1) w_0(J_2) s_4 w_0(J_2') \\ =& 123456987 \cdot 154326789 \cdot 123546789 \cdot 123765489 \\ =& 154963287 \end{aligned}$$

is a polished element.

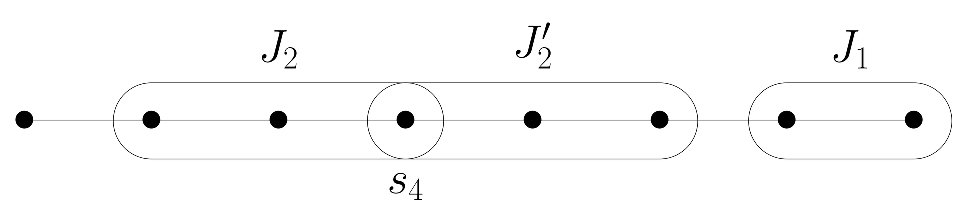

Figure 2: A polished element in type A_8

The duality map

Notice that $[e, w] \simeq [e, w'] \times [e, w'']$ if $w = w'w''$ for w' and w'' living in disjoint parabolic subgroups. So to produce the duality map, it suffices to assume that

$$w = w_0(J) w_0(J \cap J') w_0(J')$$

such that $J \cap J'$ is totally disconnected (any two simple generators commute).

This map is given by

$$u \mapsto u^{\vee} := w_0(J) u^{J'} w_0(J \cap J') \cdot u_{J'} w_0(J')$$

where $u = u^{J'} u_{J'}$ is the parabolic decomposition.

Discussion and open problems

We showed that the implication (3)⇒(4) (w being polished implies that $[e, w]$ is self-dual) holds for any finite Coxeter group, while (1)⇒(4) does not hold in general.

The following open questions are natural to ask:

- If $[e, w]$ is self-dual, is w polished?
- Are the conditions of $[e, w]$ being self-dual and w being polished characterized by pattern avoidance in the sense of Billey-Postnikov [4]?
- Is there a geometric interpretation for self-duality (even in type A)?

References

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