Separable elements in Weyl groups

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Separable permutations

A permutation $w = w_1 ... w_n$ is called **separable** if it avoids the patterns 2413 and 3142. The following characterization of separable permutations is well known:

Proposition 1

A permutation w is separable if and only if there exists an index k such that $w_1 ... w_k$ and $w_{k+1} ... w_n$ are each separable permutations and $\{w_1, ..., w_k\}$ is either $\{1, 2, ..., k\}$ or $\{n - k + 1, ..., n\}$.

Separable permutations have appeared in a variety of contexts:

- If $p_1, ..., p_n$ are real polynomials with
$$p_1(x_0) = \cdots = p_n(x_0)$$
then the permutation relating the relative order of $p_1(x_0 - \varepsilon), ..., p_n(x_0 - \varepsilon)$ to the relative order of $p_1(x_0 + \varepsilon), ..., p_n(x_0 + \varepsilon)$ is separable.
- Testing for avoidance of separable patterns in arbitrary permutations is in P, whereas the general problem is NP-complete.
- Intervals below separable permutations in weak order are the sets of linear extensions of series-parallel posets.

In this work, we generalize separable permutations to separable elements in any finite Weyl groups; some of our results are new even for the case of separable permutations.

The weak order and inversions

Throughout, Φ is a root system with positive roots Φ^+ and simple roots Δ. Unfamiliar readers can think of the "type A_{n-1}" case where:

- $\Phi = \{e_i - e_j \mid 1 \le i = j \le n\}$,
- $\Phi^+ = \{e_i - e_j \mid 1 \le i < j \le n\}$,
- $\Delta = \{e_i - e_{i+1} \mid 1 \le i \le n - 1\}$.

The **left weak order** on a Weyl group $W = W(\Phi)$ with simple reflections $s_1, ..., s_n$ has cover relations
$$w \lessdot_L s_i w$$
whenever $\ell(w) < \ell(s_i w)$ (the length ℓ is the length of the shortest expression $w = s_{i_1} \cdots s_{i_\ell}$). The **right weak order** $\le_R$ is defined analogously, but with right-multiplication by s_i. In type A_{n-1} the Weyl group W is the symmetric group and $s_i = (i\ i+1)$.

Let Φ be the root system associated to W. Then the **inversion set** is:
$$I_\Phi(w) := \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}.$$

It is a standard fact that containment of inversion sets characterizes the weak order:
$$v \le_L w \iff I_\Phi(v) \subseteq I_\Phi(w).$$

Patterns and root subsystems

Billey and Postnikov [1] introduced a notion of pattern avoidance for general root systems. We say Φ' is a **subsystem** of Φ if $\Phi' = \Phi \cap U$ for some linear subspace U of span(Φ). For $w \in W(\Phi)$, we say w **contains the pattern** (w', Φ') if
$$I_\Phi(w) \cap U = I_{\Phi'}(w')$$
and we write $w|_{\Phi'} = w'$ in this case. We say w avoids (w', Φ') if it does not contain any pattern isomorphic to (w', Φ').

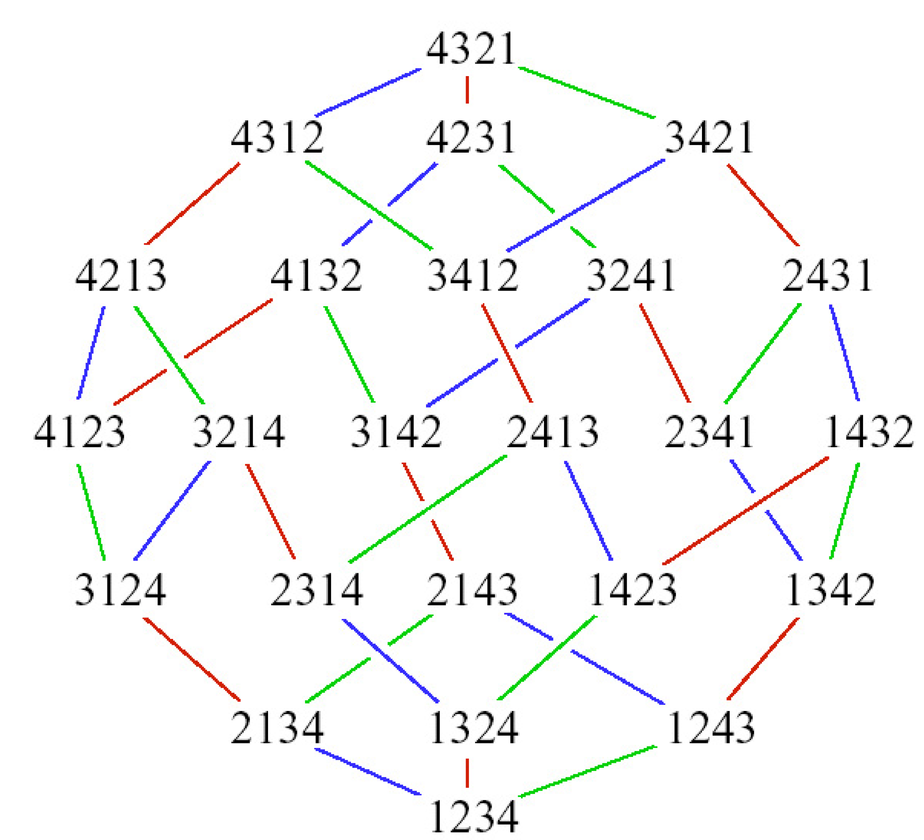

Figure 1: The left weak order on S_4. Note that the intervals below 3142 and 2413 are not rank-symmetric.

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The **root poset** is the partial order $(\Phi^+, \le)$ where $\alpha \le \beta$ if $\beta - \alpha$ is a nonnegative linear combination of elements of Δ.

Morally, the following definition is trying to replicate the recursive structure from Proposition 1 in other types.

Definition

We say an element $w \in W(\Phi)$ is *separable* if one of the following holds:

- Φ is of type A_1;
- $\Phi = \bigsqcup \Phi_i$ and $w|_{\Phi_i}$ is separable for each i;
- Φ is irreducible and there exists a *pivot* $\alpha_i \in \Delta$ such that $w|_{\Phi'} \in W(\Phi')$ is separable where Φ' is generated by $\Delta' = \Delta \setminus \{\alpha_i\}$ and such that either
$$\{\beta \in \Phi^+ : \beta \ge \alpha_i\} \subseteq I_\Phi(w), \text{ or}$$
$$\{\beta \in \Phi^+ : \beta \ge \alpha_i\} \cap I_\Phi(w) = \emptyset.$$

Results

Let Λ_w and V_w denote the order ideal and order filter generated by w in the weak order. For any poset A, let $A(q)$ denote its rank generating function. The following theorem answers an open problem of Fan Wei [2], who proved it for permutations.

Theorem 1

Let $w \in W$ be separable. Then Λ_w and V_w are rank-symmetric and rank-unimodal and
$$\Lambda_w(q) V_w(q) = W(q).$$

When $w = w_0(J)$ for $J \subseteq S$ this result recovers the well-known fact that
$$W^J(q) W_J(q) = W(q).$$

In fact, Theorem 1 can be strengthened. For X, Y any subsets of W, we say (X, Y) is a **splitting** of W if the map $X \times Y \to W$ sending $(x, y) \to xy$ is length-additive ($\ell(xy) = \ell(x) + \ell(y)$) and a bijection.

Part (b) of the following theorem answers an open problem of Björner and Wachs (1988).

Theorem 2

(a) Let $u \in W$ be separable, then $([e, u]_L, [e, u^{-1} w_0]_R)$ is a splitting of W.

(b) For $W = S_n$, these are the *only* splittings.

Theorem 2 classifies when products of the right form coming from weak order intervals are bijections; the next one states that they are always surjections.

Theorem 3

Let $W = S_n$, then for *any* $u \in W$ the map
$$([e, u]_L, [e, u^{-1} w_0]_R) \to W$$
$$(x, y) \to xy$$
is *surjective*.

After translating the above theorem into the context of linear extensions of 2-dimensional posets, we answer an open problem of Morales–Pak–Panova by giving a combinatorial proof of a theorem of Sidorenko.

Results

We also give a pattern-avoidance characterization of separable elements, which recovers the definition of separable permutations.

Theorem 4

An element $w \in W(\Phi)$ is separable if and only if it avoids the patterns:

- 3142 and 2413 of type A_3,
- the two patterns of length two in type B_2, and
- the six patterns of lengths two, three, and four in type G_2.

Theorem 5

The number of separable elements in $W(\Phi)$ is twice the number of faces in the **graph associahedron** of the Dynkin diagram of Φ.

References

[1] Sara Billey and Alexander Postnikov.
Smoothness of Schubert varieties via patterns in root subsystems.
Adv. in Appl. Math., 34(3):447–466, 2005.

[2] Fan Wei.
Product decompositions of the symmetric group induced by separable permutations.
European J. Combin., 33(4):572–582, 2012.

[3] Christian Gaetz and Yibo Gao.
Separable elements in Weyl groups.
Adv. in Appl. Math., 113:101974, 23, 2020.

[4] Christian Gaetz and Yibo Gao.
Separable elements and splittings of Weyl groups.
2019.
arXiv:1911.11172 [math.CO].

[5] Anders Björner and Michelle L. Wachs.
Generalized quotients in Coxeter groups.
Trans. Amer. Math. Soc., 308(1):1–37, 1988.

[6] Alejandro H. Morales, Igor Pak, and Greta Panova.
Why is pi less than twice phi?
Amer. Math. Monthly, 125(8):715–723, 2018.

Acknowledgements

We wish to thank Alexander Postnikov, Richard Stanley, and Anders Björner for their helpful comments. C.G. was partially supported by a National Science Foundation Graduate Research Fellowship under Grant No. 1122374.