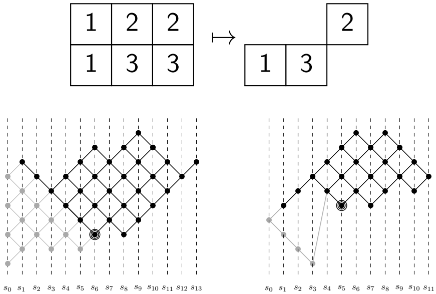
Block number and descents of fully commutative elements in B_n



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Permutation Patterns '20

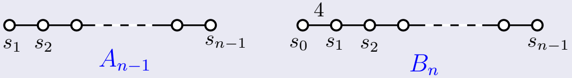
Signed permutations

For $n > 0$, we set $[n] := \{1, \ldots, n\}$ and $[\pm n] := \{\pm 1, \ldots, \pm n\}$.

Definition

We denote by S_n the symmetric group on $[n]$, and by B_n the group of signed permutations consisting of all bijections $w : [\pm n] \longrightarrow [\pm n]$, that satisfy $w(-i) = -w(i)$.

S_n and B_n are the Coxeter groups of types



Example

For $w = [4, -3, 5, 1, -2] \in B_5$ we have
$w(1) = 4,\ w(2) = -3,\ w(3) = 5,\ w(4) = 1,\ w(5) = -2.$

Fully commutative elements

Definition

An element in a Coxeter group $w \in W$ is fully commutative if any reduced expression of w can be obtained from any other one by using only commutation relations.
The set of these elements in W is denoted by $\mathrm{FC}(W)$.

Proposition (Billey-Jockush-Stanley, Stembridge)

- $\pi \in \mathrm{FC}(S_n) \iff \pi$ *avoids the pattern* 321.
- $w \in \mathrm{FC}(B_n) \iff w$ *avoids the pattern* $(-1, -2)$ *and all the patterns* (a, b, c) *such that* $|a| > b > c$ *or* $-b > |a| > c$.

Example

$$[4, -3, 5, 1, -2] \notin \mathrm{FC}(B_5) \text{ but } [-4, 5, -1, 2, 3] \in \mathrm{FC}(B_5)$$

Block number

Definition

A nonempty permutation of S_n which is not a direct sum of two nonempty permutations is called $\oplus$-irreducible.
Each permutation π can be written uniquely as a direct sum of $\oplus$-irreducible ones, called the blocks of π.

Definition

The block number of a permutation $\pi \in S_n$ is

$$\mathrm{bl}(\pi) := \#\textit{blocks of } \pi.$$

Example

Let $\pi = [2,3,1,5,4]$. Then the blocks of π are $[2,3,1]$ and $[5,4] \equiv [2,1]$, thus $\mathrm{bl}(\pi) = 2$.

Block number for type B

Definition

For $w = [w_1, \ldots, w_n] \in B_n$, let $\tau(w)$ be the permutation in S_n which records the letters $w_1, \ldots, w_n$ in the relative standard order.

Definition

We define the block number for type B of $w \in B_n$ as

$$\mathrm{bl}_B(w) := bl(\tau(w)).$$

Example

Let $w = [-4, -3, -5, 1, -2]$. Then $\tau(w) = [2, 3, 1, 5, 4]$, thus $\mathrm{bl}_B(w) = 2$.

Descent and negative sets

Definition

For $w \in B_n$, let

$$\begin{aligned} \mathrm{Des}_B(w) &:= \{0 \le i \le n-1 \mid w(i) > w(i+1)\}, \ (w(0) := 0) \\ \mathrm{Neg}(w) &:= \{1 \le i \le n \mid w(i) < 0\}, \end{aligned}$$

be the descent set and the negative set of w.

Definition (Last descent)

For $w \neq id$, set $\mathrm{Ides}_B(w) := \max \mathrm{Des}_B(w)$ *and* $\mathrm{Ides}_B(id) := 0$.

Example

$$\mathrm{Ides}_B([-3,1,-6,-2,-5,4]) = 4 \ , \qquad \mathrm{Ides}_B([-3,1,2]) = 0.$$

Equidistribution on $\mathrm{FC}(B_n)$

Theorem (BBJR, 2020)

For any $n \geq 1$, we have

$$\sum_{w \in \mathrm{FC}(B_n)} \mathbf{x}^{\mathrm{Des}_B(w)} \mathbf{z}^{\mathrm{Neg}(w)} q^{\mathrm{bl}_B(w^{-1})} t^{n-\mathrm{Ides}_B(w^{-1})} = \sum_{w \in \mathrm{FC}(B_n)} \mathbf{x}^{\mathrm{Des}_B(w)} \mathbf{z}^{\mathrm{Neg}(w)} q^{n-\mathrm{Ides}_B(w^{-1})} t^{\mathrm{bl}_B(w^{-1})}.$$

Proof Ingredients:

- An explicit description of the intersection of $\mathrm{FC}(B_n)$ with S_n-cosets in B_n.
- A type B-analogue of Rubey's descent set preserving involution $f : \mathrm{FC}(S_n) \to \mathrm{FC}(S_n)$ which sends $\mathrm{bl}(\pi)$ to $n - \mathrm{Ides}(f(\pi)^{-1})$.

Intersection of $\mathrm{FC}(B_n)$ with S_n-cosets

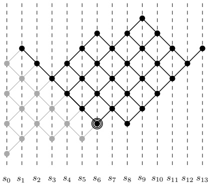
For $1 \leq i \leq n$, set $\delta_i := s_{i-1} \cdots s_1 s_0$, and

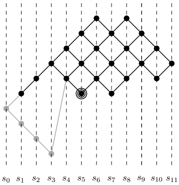
$$B_n(\pi) := \begin{cases} \{\mu \in B_n \mid \boldsymbol{\mu} = \mu_1 \cdots \mu_{v(\pi)},\ \mu_i \in \{e, \delta_i\}\} & \text{if } 1 \notin \mathrm{Des}(\pi^{-1}); \\ \{\mu \in B_n \mid \boldsymbol{\mu} \in \{e, \delta_1, \ldots, \delta_{v(\pi)}\}\} & \text{if } 1 \in \mathrm{Des}(\pi^{-1}), \end{cases}$$

where $$v(\pi) := \begin{cases} \min\left\{\mathrm{Des}(\pi^{-1}) \setminus \{1\}\right\} & \text{if } \mathrm{Des}(\pi^{-1}) \setminus \{1\} \neq \emptyset; \\ n & \text{if } \mathrm{Des}(\pi^{-1}) \setminus \{1\} = \emptyset. \end{cases}$$

Theorem (BBJR, 2020)

$$\mathrm{FC}(B_n) = \bigsqcup_{\pi \in \mathrm{FC}(S_n)} B_n(\pi) \cdot \pi$$





Chow's fundamental type B basis

The type B Chow fundamental quasi-symmetric function indexed by $J \subseteq \{0\} \cup [n-1]$ is defined as

$$F_J^B(x_0, x_1, x_2, \dots) := \sum_{\substack{0 \le i_1 \le i_2 \le \cdots \le i_n \\ j \in J \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

Problem (A type B-analogue of Sagan and Woo, '14)

Find sets of patterns Π and statistics $\mathrm{st} : B_n \to \mathbb{N} \cup \{0\}$, such that $\sum_{w \in B_n(\Pi)} q^{\mathrm{st}(w)} F^B_{\mathrm{Des}_B(w)}$ is Schur-positive.

We answer this question for $B_n(\Pi) = \mathrm{FC}(B_n)$ and $\mathrm{st} = \mathrm{bl}_B$.

Bi-tableaux

Definition

A bi-shape $\lambda := (\lambda^+, \lambda^-) \vdash n$ *is a pair of partitions of total size* n.
A bi-tableaux $T := (T^+, T^-) \in \mathrm{SYT}(\lambda)$ *is a standard filling of* λ.

Definition

The type B descent set *of a bi-tableau* T *is*

$$\mathrm{Des}_B(T) := \{i \in [n-1] : \ i+1 \text{ is in a lower row than } i\} \cup \{0 : \ 1 \in T^-\},$$

where the shape λ^- *is drawn southwest of the shape* λ^+.
Let $\mathrm{Ides}_B(T) := \max \mathrm{Des}_B(T)$.

Example

$$T = \begin{array}{ccccc} & & & \boxed{1} & \boxed{3} \\ \boxed{2} & \boxed{4} & \boxed{5} & & \end{array} \in \mathrm{SYT}((2),(3)), \quad \mathrm{Ides}_B(T) = 3.$$

Main result

Theorem (BBJR, 2020)

For any positive integer n, we have

$$\sum_{w \in \mathrm{FC}(B_n) \setminus \mathrm{FC}(S_n)} q^{\mathrm{bl}_B(w^{-1})} F^B_{\mathrm{Des}_B(w)}(x_0, x_1, x_2, \ldots)$$

$$= \sum_{k=1}^{n} \left(\sum_{j=0}^{n} b_{n,k,j} q^j \right) s_{(k)}(x_0, x_1, x_2, \ldots)\ s_{(n-k)}(x_1, x_2, \ldots),$$

where $s_{(k)}$ is the Schur function indexed by the partition (k), and

$$b_{n,k,j} := \#\{T \in \mathrm{SYT}((n-k),(k)) : \ \mathrm{Ides}_B(T) = n - j\} \geq 0.$$

Proof Ingredients:

- The description of $\mathrm{FC}(B_n)$ as a disjoint union of two-sided Kazhdan–Lusztig cells, given by Green and Losonczy.
- A descent set preserving bijection from domino tableaux to bi-tableaux, expanding upon Barbash–Vogan's bijection.
- The above equidistribution theorem.

Thanks for the attention,
See you soon!