Enumeration with Moon Polyominoes and Beyond

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Outline

1. Two interesting Problems

Permutation Statistics

Let $\sigma = a_1 a_2 \cdots a_n$ be a permutation,

- $\mathrm{Inv}(\sigma) = \{(a_i, a_j) : \ i < j, \ a_i > a_j\}$, $\quad \mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)|$. and $\mathrm{coinv}(\sigma) = \binom{n}{2} - \mathrm{inv}(\sigma)$.

Permutation Statistics

Let $\sigma = a_1 a_2 \cdots a_n$ be a permutation, e.g. $\sigma = 3712645$

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  and $\mathrm{coinv}(\sigma) = \binom{n}{2} - \mathrm{inv}(\sigma)$.
  $\mathrm{Inv}(\sigma) = \{31, 32, 71, 72, 76, 74, 75, 64, 65\}$,
  $\mathrm{inv}(\sigma) = 9$, $\mathrm{coinv}(\sigma) = 12$

Permutation Statistics

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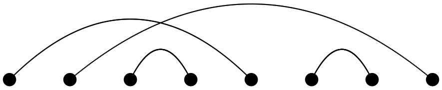
  and $\mathrm{coinv}(\sigma) = \binom{n}{2} - \mathrm{inv}(\sigma)$.
  $\mathrm{Inv}(\sigma) = \{31, 32, 71, 72, 76, 74, 75, 64, 65\}$,
  $\mathrm{inv}(\sigma) = 9$, $\mathrm{coinv}(\sigma) = 12$

Theorem

$$\sum_{\pi \in \mathfrak{S}_n} p^{\mathrm{inv}(\pi)} q^{\mathrm{coinv}(\pi)} = [n]_{p,q}!$$

where $[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}$.

Matchings: crossings and nestings



$$\mathrm{cr}_2(\alpha) = 1, \qquad \mathrm{ne}_2(\alpha) = 3.$$

Let

$$L_n(p,q) = \sum_{\alpha \in M(2n)} p^{\mathrm{cr}_2(\alpha)} q^{\mathrm{ne}_2(\alpha)},$$

Theorem (Touchard, Riordan, Kasraoui & Zeng)

$$\sum_{n \geq 0} L_n(p,q) z^n = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{[2]_{p,q} z}{1 - \cfrac{[3]_{p,q} z}{\cdots}}}}$$

If we fix the sets of minimal elements and maximal elements,

$$\sum_M p^{\mathrm{cr}_2(M)} q^{\mathrm{ne}_2(M)} = (p+q)(p+q) = [2]_{p,q}[2]_{p,q}.$$

In general, it is always a product of (p,q)-integers, where $[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}$.

Problem 1.

A crossing/nesting with edges (i_1, j_1), (i_2, j_2) is:

$$\begin{cases} \text{odd} & \text{if } i_1 \text{ is odd} \\ \text{even} & \text{if } i_1 \text{ is even} \end{cases}$$

Observation

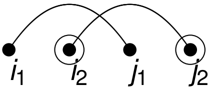
$$\sum_M p^{\text{odd cr}_2 + \text{even ne}_2} q^{\text{even cr}_2 + \text{odd ne}_2} = \sum_M p^{\text{cr}_2(M)} q^{\text{ne}_2(M)}$$

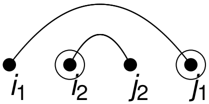
Problem 2.

A crossing with edges (i_1, j_1), (i_2, j_2) is:
$$\begin{cases} \text{small} & \text{if } i_2 + j_2 \leq 2n; \\ \text{large} & \text{otherwise} \end{cases}$$

A nesting with edges (i_1, j_1), (i_2, j_2) is $\begin{cases} \text{small} & \text{if } i_2 + j_1 \leq 2n; \\ \text{large} & \text{otherwise} \end{cases}$

Back





Observation

$$\sum_M p^{\text{small } \mathrm{cr}_2 + \text{large } \mathrm{ne}_2} q^{\text{large } \mathrm{cr}_2 + \text{small } \mathrm{ne}_2} = \sum_M p^{\mathrm{cr}_2(M)} q^{\mathrm{ne}_2(M)}$$

2. A Combinatorial Model

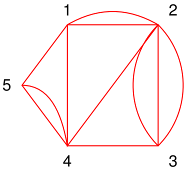
Fillings of polyominoes

Polyomino: a finite subset of $\mathbb{Z}^2$, represented by square cells; each cell is assigned a natural number.

- 01-fillings of rectangles: Permutations and Words
- Triangular shape: graphs on $[n]$

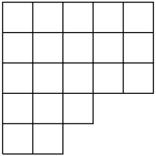
e.g. $\sigma = 4132$

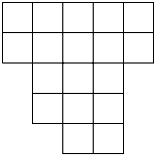
	1	2	3	4
1				1
2	1			
3			1	
4		1		



	5	4	3	2
1	1	1	0	2
2	0	1	3	
3	0	1		
4	2			

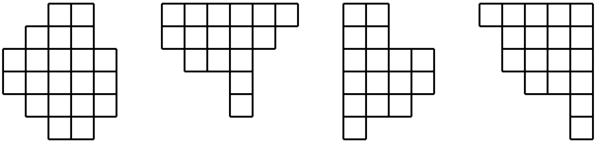
- Ferrers diagrams (Backelin, West & Xin; Krattenthaler, de Mier): Graphs with given degree sequence, matchings, set partitions with given MIN/ MAX block-elements, rook placements
- Stack polyominoes: k-triangulartions (Jonsson; Jonsson & Welker); pattern avoidance of set partitions (Jelínek & Mansour); Hecke insertion (Guo and Poznanovic)





Moon polyominoes

convex any column or row is connected.
intersection-free Every two rows are comparable, i.e., the column-coordinates of the longer one cover those of the shorter one.
moon polyomino (L-connected) a convex and intersection-free polyomino (Rubey, Kasraoui,Yan, ...)



We consider 01-fillings.

A combinatorial hierarchy

permutation/word $\subseteq$ matching/set partitions $\subseteq$ graphs
$\subseteq$ Ferrers diagrams $\subseteq$ stack polyominoes
$\subseteq$ moon polyominoes
$\subseteq$ more general polyominoes

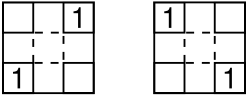
Allow various approaches and techniques, e.g.

- fix the polyomino and change the fillings
- transform the polyominoes
- bijection, induction,
- tableaux operations ...

3. Chains of length 2

Basic correspondence

	inversion/coinversion	in	permutations
$\Longleftrightarrow$	crossings/nestings of two edges	in	matchings
$\Longleftrightarrow$	northeast/southeast chains of size 2	in	01-fillings



Denote by $\mathrm{ne}_2(M)$ and $\mathrm{se}_2(M)$ the numbers of ne/se chains of size 2 in a filling M.

A unified theorem

Given $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{N}^n$ and $\mathbf{e} = (e_1, \ldots, e_m) \in \{0,1\}^m$. Let $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$ be the set of 01-fillings of $\mathcal{M}$ with row-sum $\mathbf{s}$ and column-sum $\mathbf{e}$.

Theorem (Kasraoui 2010)

$$\sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{\mathrm{ne}_2(M)} q^{\mathrm{se}_2(M)} = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} q^{\mathrm{ne}_2(M)} p^{\mathrm{se}_2(M)} = \prod_{i=1}^{n} \begin{bmatrix} h_i \\ s_i \end{bmatrix}_{p,q}$$

It contains results of

- permutations: $\sum_{\pi \in \mathfrak{S}_n} p^{\mathrm{inv}(\pi)} q^{\mathrm{coinv}(\pi)} = [n]_{p,q}!$.
- matchings [de Sainte-Catherine]
- set partitions [Kasraoui & Zeng]
- crossings and alignments for permutation [Corteel]
- linked partitions [Chen, Wu & Y]

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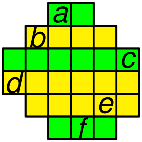
REMARK

1. Not true if allow arbitrary row sum *and* arbitrary column sum
2. Not true if not *convex* or *intersection-free*

A mixed variant

Bicolor the rows of $\mathcal{M}$ and mix the 2-chains by the position of the top cell.



Define top-mixed statistics

$\alpha(M)$: 

or 

$\beta(M)$: 

or 

$\alpha(M) = 3$ with chains *ef*, *ae*, *af*;
$\beta(M) = 3$ with chains *be*, *cd*, *ce*.

Over $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$

Theorem (Chen, Wang, Y & Zhao)

The joint distribution $(\alpha(M), \beta(M))$ is always symmetric and independent of the bi-coloring. In particular,

$$\sum_M p^{\alpha(M)} q^{\beta(M)} = \sum_M p^{\mathrm{ne}_2(M)} q^{\mathrm{se}_2(M)}$$

Also true if one mixes by the bottom cell, or bi-coloring columns of $\mathcal{M}$.

This explains Problem 1. ◂ Problem 1

Charged polyominoes

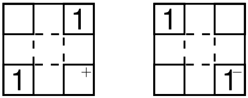
Equip with $\mathcal{M}$ a charge function $C : \mathcal{M} \to \{\pm 1\}$.
For a 2×2 submatrix S of $\mathcal{M}$, set $\operatorname{sgn}(S)$ as the charge of its lower-right corner.

Definition

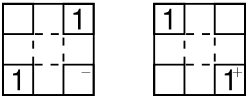
A chain with the support matrix S is ***positive*** *with respect to C if it is*

1. *a northeast chain with* $\operatorname{sgn}(S) = 1$, *or*
2. *a southeast chain with* $\operatorname{sgn}(S) = -1$.

Otherwise, the chain is negative.



(a) Positive chains



(b) Negative chains

Conjecture

The distribution of $(\mathrm{pos}_C(M), \mathrm{neg}_C(M))$ *does not depend on the charge function* C. *Consequently,*

$$\sum_{M \in \mathbf{F}(\mathcal{M},\mathbf{s},\mathbf{e})} p^{\mathrm{pos}_C(M)} q^{\mathrm{neg}_C(M)} = \sum_{M \in \mathbf{F}(\mathcal{M},\mathbf{s},\mathbf{e})} p^{\mathrm{ne}_2(M)} q^{\mathrm{se}_2(M)}.$$

Not always!

Theorem (Wang & Y)

The conjecture is true if the polyomino is top aligned or left aligned.

This explains Problem 2.

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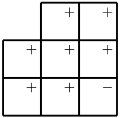
In general,

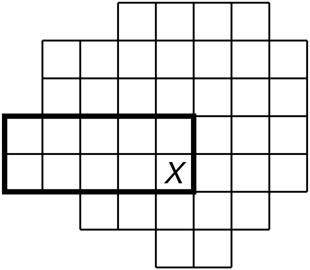


$$\sum_M p^{\mathrm{ne}_2(M)} q^{\mathrm{se}_2(M)} = p^2 + 2pq + q^2,$$

$$\sum_M p^{\mathrm{pos}_C(M)} q^{\mathrm{neg}_C(M)} = 2p^2 + 2q^2.$$

The restrictive positivity

For a cell $X \in \mathcal{M}$, define $\mathcal{R}_X$, *the box of X*, to be the widest rectangle contained in $\mathcal{M}$ whose lower right corner is X.



A 2×2 submatrix is *restrictive* if it is contained in the box of its lower right corner.

Definition

Let M be a 01-filling of a charged moon polyomino $\mathcal{M}$ with a charge function C. A chain with the support matrix S is ***restrictively positive*** *with respect to C if it is*

1. *a northeast chain with $\operatorname{sgn}(S) = 1$,*
2. *a northeast chain with $\operatorname{sgn}(S) = -1$ and* ***is not restrictive****,*
3. *a southeast chain with $\operatorname{sgn}(S) = -1$* ***and restrictive****.*

Otherwise, the chain is restrictively negative.

Symmetry in the general case

Let $\overline{\text{pos}}_C(M)$ and $\overline{\text{neg}}_C(M)$ be the numbers of restrictively positive chains and restrictively negative chains of M with respect to C. Set

$$\overline{F}_C(p,q) = \sum_{M \in \mathbf{F}(\mathcal{M},\mathbf{s},\mathbf{e})} p^{\overline{\text{pos}}_C(M)} q^{\overline{\text{neg}}_C(M)}.$$

Theorem (Wang & Y)

The bi-variate generating function $\overline{F}_C(p,q)$ does not depend on the charge function C. Consequently,

$$\overline{F}_C(p,q) = \overline{F}_+(p,q) = \sum_{M \in \mathbf{F}(\mathcal{M},\mathbf{s},\mathbf{e})} p^{\text{ne}_2(M)} q^{\text{se}_2(M)}.$$

How to work with this model:

1. Check the result in a rectangle shape: establish a bijection between fillings while changing the color of one row, or the charge of one cell.
2. In general polyominoes, gradually change the shape/coloring/sign.
3. We can also define other permutation statistics and transformations to fillings of moon polyominoes, e.g., descent and major index.

4. More General Shapes?

Motivation: in several examples, the distribution (of a combinatorial statistic) remains the same for two different polyominoes $\mathcal{M}$ and $\mathcal{M}'$, where $\mathcal{M}$ is obtained from $\mathcal{M}'$ by a permutation of rows, and both are moon polyominoes.

What happens if one swaps the rows of the polyomino?

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Idea: extend to polyominoes that allow arbitrary permutations of rows.

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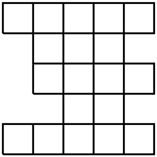
Idea: extend to polyominoes that allow arbitrary permutations of rows.

Layer Polyomino: intersection-free and row-convex, but not necessarily column-convex.

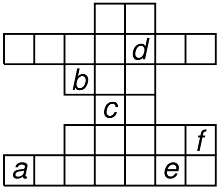


Regular chains in fillings of layer polyomino

A 2×2 submatrix

$$S = \{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2) \in \mathcal{L} : i_1 < i_2, j_1 < j_2\}.$$

ne-chain: S with $(i_1, j_2), (i_2, j_1)$ filled with 1.
se-chain: S with $(i_1, j_1), (i_2, j_2)$ filled with 1.



4 ne-chains: bd, cd, ad, ef 2 se-chains: df, de

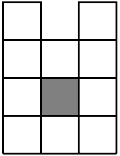
Main result on Layer Polyomino $\mathcal{L}$

Fix row sum **s** and column sum **e**.

Theorem (Phillipson, Y & Yeh)

1. *If either* **s** *or* **e** *is a 01-vector, then permuting rows of* $\mathcal{L}$ *does not change the distribution of* $(\mathrm{ne}_2, \mathrm{se}_2)$. *Consequently, the distribution of* $(\mathrm{ne}_2, \mathrm{se}_2)$ *is symmetric.*
2. *For arbitrary* **s**, **e**, *the distribution of* $(\mathrm{ne}_2, \mathrm{se}_2)$ *may not be symmetric.*

Cannot get rid of all convexity.



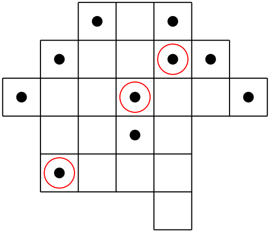
5. Size of the longest chains

Patterns $12\cdots k$ and $k(k-1)\cdots 21$

1. Permutation: sizes of the longest increasing/decreasing subsequences
2. Matchings and Set Partitions: sizes of the largest crossing/nesting (Chen, Deng, Du, Stanley & Y'07)
3. Ferrers Diagrams: sizes of maximal ne/se chains (Backelin, West & Xin'07, Kratthenthaler'06, de Mier'06)
4. Stack Polyomino, k-triangulation: avoiding $(k+1)$-ne chains (Jonsson'07, Jonsson & Welker'07)
5. Moon Polyominoes: for both 01 and $\mathbb{N}$ fillings, (Rubey'11)

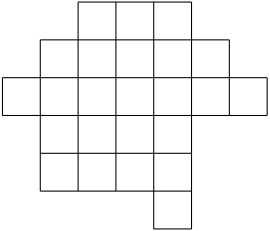
Statistic $\mathrm{ne}(M)$, the size of maximal northeast chains?

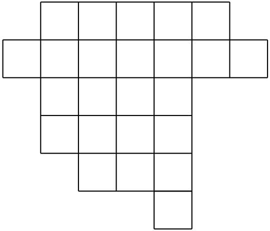
- A k-ne-chain: A set of 1-cells $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ with $i_1 < \cdots < i_k, j_1 < \cdots < j_k$ such that the smallest rectangle containing them is a subset of $\mathcal{M}$.
- For a 01-filling M, $\mathrm{ne}(M)$ is the length of the largest ne-chain.



Permuting rows

For a moon polyomino $\mathcal{M}$, let $\sigma\mathcal{M}$ be another moon polyomino obtained by permuting the rows of M.





Theorem (Rubey)

In 01-fillings with given column sum, ne *has the same distribution over fillings of* $\mathcal{M}$ *and* $\sigma\mathcal{M}$*.*

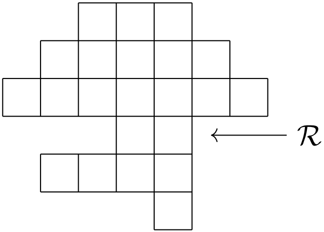
Bijection in moon polyominoes are complicated.

We hope to change the shape of the polyomino only slightly.

Unfornately, exchanging rows in layer polyominoes DOES NOT preserve the distribution of $\mathrm{ne}(M)$.

Almost-Moon Polyominoes

- *comparable* rows and columns
- *convex* rows
- at most one exceptional row



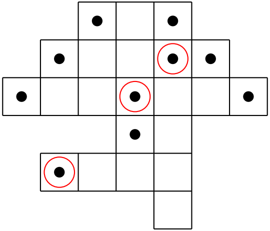
- A row $\mathcal{R}$ is an *exceptional row* of a polyomino $\mathcal{M}$ if there are *longer* rows both above $\mathcal{R}$ and below $\mathcal{R}$.

ne-chains in fillings of almost-moon polyominoes

- A k-ne-chain is a set of k cells $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ with $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$ filled with 1's such that the $k \times k$ submatrix

$$\{(i_r, j_s) : 1 \leq r \leq k, 1 \leq s \leq k\}$$

is contained in the polyomino
(with no restriction on the filling of the other cells).



A bijection ϕ between almost-moon polyominoes

- $\mathbf{F}(\mathcal{M})$ = the set of all 01-fillings of a polyomino $\mathcal{M}$

$\mathcal{M}$ and $\mathcal{N}$: almost-moon polyominoes obtained by swapping two adjacent rows.

Theorem (Poznanovic & Yan)

There is a bijection

$$\phi_{\mathcal{M},\mathcal{N}} : \mathbf{F}(\mathcal{M}) \longrightarrow \mathbf{F}(\mathcal{N})$$

that preserves the column sums of the fillings and such that

$$\mathrm{ne}(\phi_{\mathcal{M},\mathcal{N}}(M)) = \mathrm{ne}(M) \quad \textit{for} \quad M \in \mathbf{F}(\mathcal{M}).$$

Main Idea of the construction

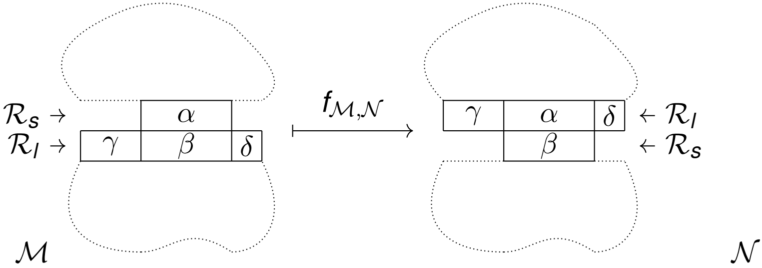
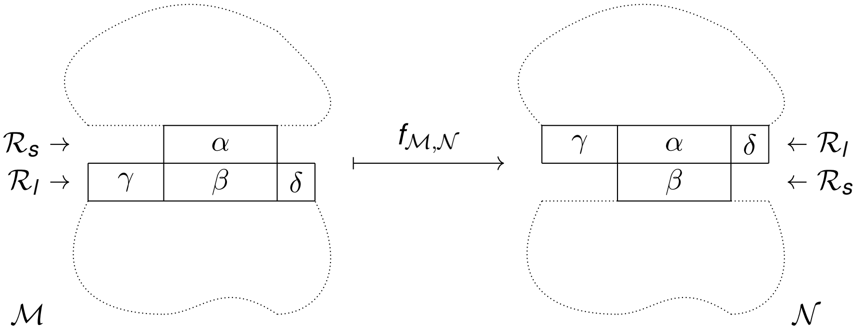


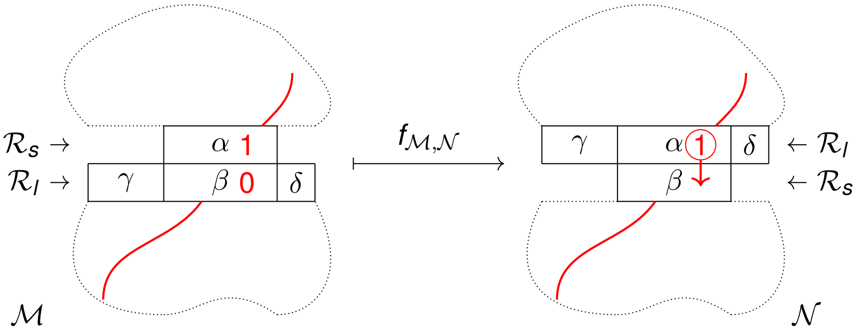
Figure: The fillings M and $f_{\mathcal{M},\mathcal{N}}(M)$ differ only in the rows $\mathcal{R}_s$ and $\mathcal{R}_l$.

Construction 2



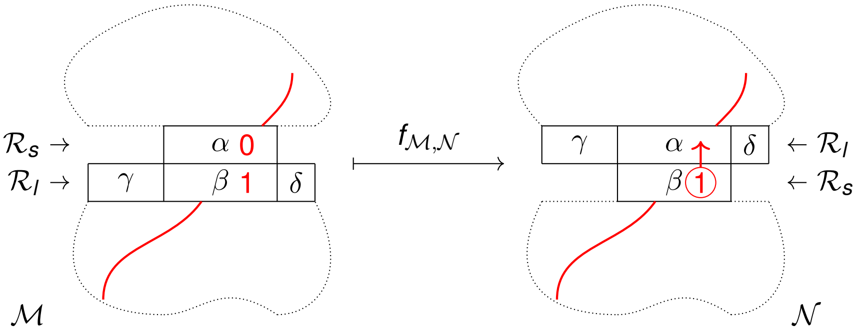
- if $\mathrm{ne}(f(M)) = \mathrm{ne}(M)$ then $\phi_{\mathcal{M},\mathcal{N}}(M) = f_{\mathcal{M},\mathcal{N}}(M)$.

Construction 2



- If $\mathrm{ne}(M) = k$ and $\mathrm{ne}(f(M)) = k + 1$.

Construction 2



- If $\mathrm{ne}(M) = k$ and $\mathrm{ne}(f(M)) = k - 1$:

Note

- $\mathcal{N} = \sigma\mathcal{M}$ then $\mathcal{N}$ can be obtained from $\mathcal{M}$ by several interchanges of adjacent rows so that the intermediate polyominoes are also almost-moon.
- Composing the maps from our theorem, we get a bijection that proves Rubey's result for moon polyominoes.
- Our map preserves the column sums, but not the row sums (not even if they are 1).

On restricted 01-fillings

- $\mathbf{F}(M, \mathbf{r}, \mathbf{c})$ = the set of 01-fillings of a polyomino $\mathcal{M}$ with row-sum vector $\mathbf{r}$ and column-sum vector $\mathbf{c}$

$\mathcal{M}$ and $\mathcal{N}$ same as before.

Theorem (Poznanovikj & Yan)

If $\boldsymbol{r} \in \{0,1\}^*$, then there is a bijection

$$\psi_{\mathcal{M},\mathcal{N}} : \mathbf{F}(\mathcal{M}, \boldsymbol{r}, \boldsymbol{c}) \longrightarrow \mathbf{F}(\mathcal{N}, \sigma\boldsymbol{r}, \boldsymbol{c})$$

such that

$$\mathrm{ne}(\psi_{\mathcal{M},\mathcal{N}}(M)) = \mathrm{ne}(M) \quad \textit{for} \quad M \in \mathbf{F}(\mathcal{M}, \boldsymbol{r}, \boldsymbol{c}).$$

Open questions

- Are there a representation theory explanation for the equi-distribution of various combination of 2-chains?
- How about other patterns, in particular, k-chains?
- In studying the maximal ne/se chains, there are three main approaches: combinatorial transformations and bijections, operations on tableaux, and simplicial complex and commutative algebra. What is the big picture behind them?
- Is there a "nice" bijection that reverses the sequence of the rows of the polyomino?

THANK YOU FOR YOUR ATTENTION

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