Bijective proofs of shuffle compatibility

by

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Define a permutation to be any sequence of distinct positive integers. Given two permutations π and σ on disjoint underlying sets, we denote by $\pi \sqcup\!\sqcup \sigma$ the set of shuffles of π and σ. A statistic is a function St whose domain is the set of permutations. A statistic is shuffle compatible if the distribution of St on $\pi \sqcup\!\sqcup \sigma$ depends only on $\mathrm{St}(\pi)$ and $\mathrm{St}(\sigma)$ and their lengths rather than on the individual permutations themselves. This notion is implicit in the work of Stanley in his theory of P-partitions. The definition was explicitly given by Gessel and Zhuang who proved that various permutation statistics were shuffle compatible using mainly algebraic means. This work was continued by Grinberg and by Oğuz. We present bijective demonstrations of shuffle compatibility. In particular, a large number of permutation statistics can be shown to be shuffle compatible using a few simple bijections. Our approach also leads to a method for constructing such bijective proofs rather than having to treat each one in an ad hoc manner. Finally, we are able to prove a conjecture of Gessel and Zhuang about the shuffle compatibility of a certain statistic.