Fully commutative elements in the Coxeter group of type $A_{n-1}$ are in one-to-one correspondence with 321-avoiding permutations in the symmetric group $S_n$, as shown in the classical paper [3]. The set of fully commutative elements in type $B_n$, denoted by $\text{FC}(B_n)$, has an explicit combinatorial description in terms of forbidden patterns in signed permutations [11].

The block number of a permutation $\pi$ in $S_n$, which was studied in [9] as the cardinality of the connectivity set of $\pi$, is equal to the maximal number of summands in an expression of $\pi$ as a direct sum of smaller permutations. It was shown recently in [1] that the quasi-symmetric generating function of the descent set statistic over the set of 321-avoiding permutations with prescribed block number is Schur-positive.

In the current paper, Chow’s quasi-symmetric function determined by the set of fully commutative elements in type $B_n$ [5], with appropriate block number and descent set, is shown to be symmetric and Schur-positive. An explicit combinatorial description of the coefficients in the Schur expansion is provided.

The proof involves several ingredients:

(i) The description of $\text{FC}(B_n)$ as a disjoint union of two-sided Kazhdan–Lusztig cells, given by Green and Losonczy [6].

(ii) A recent descent set preserving bijection from domino tableaux to bi-tableaux, which expands upon Barbash–Vogan’s bijection [4] and [7, 12].

(iii) A type $B$ extension of Rubey’s involution [8] on 321-avoiding permutations. The latter is in particular descent set preserving and maps the block number to the last descent position of the inverse.

(iv) A detailed study of the intersection of $\text{FC}(B_n)$ with $S_n$-cosets. This is done by using the descriptions of fully commutative elements in types $A_{n-1}$ and $B_n$ in terms of heaps, which were given for instance in [2, 10].

References