BLOCK NUMBER AND DESCENTS OF FULLY COMMUTATIVE ELEMENTS IN B_n

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Fully commutative elements in the Coxeter group of type A_{n-1} are in one-to-one correspondence with 321-avoiding permutations in the symmetric group S_n, as shown in the classical paper [3]. The set of fully commutative elements in type B_n, denoted by $\mathrm{FC}(B_n)$, has an explicit combinatorial description in terms of forbidden patterns in signed permutations [11].

The block number of a permutation π in S_n, which was studied in [9] as the cardinality of the connectivity set of π, is equal to the maximal number of summands in an expression of π as a direct sum of smaller permutations. It was shown recently in [1] that the quasi-symmetric generating function of the descent set statistic over the set of 321-avoiding permutations with prescribed block number is Schur-positive.

In the current paper, Chow's quasi-symmetric function determined by the set of fully commutative elements in type B_n [5], with appropriate block number and descent set, is shown to be symmetric and Schur-positive. An explicit combinatorial description of the coefficients in the Schur expansion is provided.

The proof involves several ingredients:

(i) The description of $\mathrm{FC}(B_n)$ as a disjoint union of two-sided Kazhdan–Lusztig cells, given by Green and Losonczy [6].
(ii) A recent descent set preserving bijection from domino tableaux to bi-tableaux, which expands upon Barbash–Vogan's bijection [4] and [7, 12].
(iii) A type B extension of Rubey's involution [8] on 321-avoiding permutations. The latter is in particular descent set preserving and maps the block number to the last descent position of the inverse.
(iv) A detailed study of the intersection of $\mathrm{FC}(B_n)$ with S_n-cosets. This is done by using the descriptions of fully commutative elements in types A_{n-1} and B_n in terms of heaps, which were given for instance in [2, 10].

References

[1] R. M. Adin, E. Bagno and Y. Roichman. *Block decomposition and Schur-positivity*, J. Algeb. Combin. **47**, (2018), 603–622.
[2] R. Biagioli, F. Jouhet and P. Nadeau. *Fully commutative elements in affine and finite Coxeter groups*, Monats. für Mathematik **178**, (2015), 1–37.
[3] S. Billey, W. Jockusch and R. P. Stanley. *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combinatorics **2** (1993), 345–374.
[4] D. Barbash and D. Vogan. *Primitive ideals and orbital integrals in complex classical groups*, Math. Ann. **259**, (1982), 153–199.
[5] C. -O. Chow. Noncommutative symmetric functions of type B. Ph.D. Thesis, Massachusetts Institute of Technology, 2001.
[6] R. M. Green and J. Losonczy. *Fully commutative Kazhdan-Lusztig cells*, Ann. Inst. Fourier **51** (2001), 1025–1045.
[7] A. Mayorova and E. Vassilieva. *A new link between the descent algebra of type B, domino tableaux and Chow's quasisymmetric functions*. Discrete Math. **342**, (2019), 1658–1673.
[8] M. Rubey. *An involution on Dyck paths that preserves the rise composition and interchanges the number of returns and the position of the first double fall*, Sém. Lothar. Combin. **77** ([2016-2018]), Art. B77f, 4 pp.
[9] R. P. Stanley. *The descent set and connectivity set of a permutation*, J. Integer Sequences **8** (2005), Art. 05.3.8, 9 pages.
[10] J. R. Stembridge. *On the fully commutative elements of Coxeter groups*, J. Algeb. Combin. **5** (1996), 353–385.
[11] J. R. Stembridge. *Some combinatorial aspects of reduced words in finite Coxeter groups*, Trans. Amer. Math. Soc. **349** (1997), 1285–1332.
[12] M. Taskin. *Plactic relations for r-domino tableaux*, Elect. J. Combin. **19** (2012), #P38

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