

Homomorphisms on Noncommutative Symmetric Functions and Permutation Enumeration

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Partially based on joint work with [Ira M. Gessel](#).

Outline

- 1 Introduction
- 2 Three Basic Homomorphisms
- 3 Homomorphisms Arising from Shuffle-Compatibility

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Permutations and Descents

- A **permutation** of **length** n (or, n -**permutation**) $\pi = \pi_1\pi_2 \cdots \pi_n$ is a linear ordering of $[n] := \{1, 2, \dots, n\}$.

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- The set of n -permutations is denoted \mathfrak{S}_n .
- We say that $k \in [n - 1]$ is a **descent** of $\pi \in \mathfrak{S}_n$ if $\pi_k > \pi_{k+1}$.
The **descent number** $\text{des}(\pi)$ is the number of descents of π .

Increasing Runs and Descent Compositions

- Descents separate permutations into **increasing runs**: maximal increasing consecutive subsequences.
- Call the tuple of increasing run lengths of π the **descent composition** of π , denoted $\text{Comp}(\pi)$.

Example

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- The notions of descents and increasing runs extend to **words** on any totally ordered alphabet (such as the positive integers \mathbb{P}).

Example

The increasing runs of the word $w = 11526249$ are **115**, **26**, and **249**, so $\text{Comp}(w) = (3, 2, 3)$ and $\text{des}(w) = 2$.

Ribbon Functions

- Let X_1, X_2, \dots be noncommuting variables. Given a composition $L = (L_1, L_2, \dots, L_k)$, define the **ribbon function** r_L by

$$r_L := \sum_{(i_1, \dots, i_n)} X_{i_1} X_{i_2} \cdots X_{i_n}$$

over all (i_1, \dots, i_n) satisfying

$$\underbrace{i_1 \leq \cdots \leq i_{L_1}}_{L_1} > \underbrace{i_{L_1+1} \leq \cdots \leq i_{L_1+L_2}}_{L_2} > \cdots > \underbrace{i_{L_1+\cdots+L_{k-1}+1} \leq \cdots \leq i_n}_{L_k}.$$

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Example

The words 221552 and 374443 have descent composition $(2, 3, 1)$, so $X_2^2 X_1 X_5^2 X_2$ and $X_3 X_7 X_4^3 X_3$ are both terms in $r_{(2,3,1)}$.

Noncommutative Symmetric Functions

- Let \mathbf{Sym}_n be the vector space with basis $\{\mathbf{r}_L\}_{L \models n}$. Then

$$\mathbf{Sym} := \bigoplus_{n=0}^{\infty} \mathbf{Sym}_n$$

is a subalgebra of $\mathbb{Q}\langle\langle X_1, X_2, \dots \rangle\rangle$ called the **algebra of noncommutative symmetric functions**.

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- Noncommutative symmetric functions were formally introduced in 1995 by Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon, but appeared implicitly in the 1977 Ph.D. thesis of Ira Gessel.

Noncommutative Symmetric Functions

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- Noncommutative symmetric functions were formally introduced in 1995 by Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon, but appeared implicitly in the 1977 Ph.D. thesis of Ira Gessel.
- Let $\mathbf{h}_n := \mathbf{r}_{(n)} = \sum_{i_1 \leq i_2 \leq \dots \leq i_n} X_{i_1} X_{i_2} \cdots X_{i_n}$. Then the \mathbf{h}_n are algebraically independent and generate \mathbf{Sym} .

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Our Approach

- Many permutation enumeration formulas involving runs or descents can be proven in the following way:
 - ① Derive a lifting of the formula in **Sym**.
 - ② Apply an appropriate homomorphism.

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 - ① Derive a lifting of the formula in **Sym**.
 - ② Apply an appropriate homomorphism.
- The simplest such homomorphism is $\Phi: \mathbf{Sym} \rightarrow \mathbb{Q}[[x]]$ defined by

$$\Phi(\mathbf{h}_n) = \frac{x^n}{n!}.$$

Then

$$\Phi(\mathbf{r}_L) = \beta(L) \frac{x^n}{n!}$$

where $\beta(L)$ is the number of permutations with descent composition $L \vDash n$.

David and Barton's Formula

Lemma (Gessel & Z. 2014)

$$\sum_{\substack{L \\ \text{all parts} < m}} \mathbf{r}_L = \left[\sum_{n=0}^{\infty} (\mathbf{h}_{mn} - \mathbf{h}_{m(n+1)}) \right]^{-1}$$

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- By applying the homomorphism Φ , we get:

Theorem (David & Barton 1962)

Let $a_{n,m}$ be the number of n -permutations with every increasing run having length less than m (i.e., avoiding the consecutive pattern $12 \cdots m$). Then

$$\sum_{n=0}^{\infty} a_{n,m} \frac{x^n}{n!} = \left[\sum_{n=0}^{\infty} \left(\frac{x^{mn}}{(mn)!} - \frac{x^{mn+1}}{(mn+1)!} \right) \right]^{-1}.$$

Alternating Analogues

- We say that $k \in [n - 1]$ is an **alternating descent** of $\pi \in \mathfrak{S}_n$ if k is an odd descent or an even ascent.

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The alternating descents of $\pi = 17645823$ are 3 and 4, and the alternating runs of π are 176, 4, and 5823.

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- The **alternating descent number** and **alternating descent composition** are defined analogously.

An Alternating Analogue of Φ

- The n th Euler number E_n is defined by

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec(x) + \tan(x).$$

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$$\hat{\Phi}(\mathbf{h}_n) = E_n \frac{x^n}{n!}.$$

Then

$$\hat{\Phi}(\mathbf{r}_L) = \hat{\beta}(L) \frac{x^n}{n!}$$

where $\hat{\beta}(L)$ is the number of permutations with alternating descent composition $L \vDash n$.

An Alternating Analogue of David and Barton's Formula

- Applying $\hat{\Phi}$ to $\sum_{\substack{L \\ \text{all parts} < m}} r_L = \left[\sum_{n=0}^{\infty} (\mathbf{h}_{mn} - \mathbf{h}_{mn+1}) \right]^{-1}$ yields:

Theorem (Gessel & Z. 2014)

Let $\hat{a}_{n,m}$ be the number of n -permutations with every alternating run having length less than m . Then

$$\sum_{n=0}^{\infty} \hat{a}_{n,m} \frac{x^n}{n!} = \left[\sum_{n=0}^{\infty} \left(E_{mn} \frac{x^{mn}}{(mn)!} - E_{mn+1} \frac{x^{mn+1}}{(mn+1)!} \right) \right]^{-1}.$$

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- The case $m = 3$ counts permutations with **all peaks odd and all valleys even**, answering a question of Liviu Nicolaescu.

A q -Analogue of Φ

- We say that $(i, j) \in [n]^2$ is an **inversion** of $\pi \in \mathfrak{S}_n$ if $i < j$ and $\pi_i > \pi_j$.
- The **inversion number** $\text{inv}(\pi)$ is the number of inversions of π .

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- Define $\Phi_q: \mathbf{Sym} \rightarrow \mathbb{Q}[[q, x]]$ by

$$\Phi_q(\mathbf{h}_n) = \frac{x^n}{[n]_q!}.$$

Then

$$\Phi_q(\mathbf{r}_L) = \beta_q(L) \frac{x^n}{[n]_q!}$$

where

$$\beta_q(L) := \sum_{\pi \in \text{Comp}(L)} q^{\text{inv}(\pi)}.$$

A q -Analogue of David and Barton's Formula

- By applying the homomorphism Φ_q to

$$\sum_{\substack{L \\ \text{all parts} < m}} r_L = \left[\sum_{n=0}^{\infty} (\mathbf{h}_{mn} - \mathbf{h}_{mn+1}) \right]^{-1},$$

we get:

Theorem (Elizalde 2016)

Let $a_{n,m}(q) = \sum_{\pi} q^{\text{inv}(\pi)}$ over all n -permutations with every increasing run having length less than m (i.e., avoiding the consecutive pattern $\underline{12 \cdots m}$). Then

$$\sum_{n=0}^{\infty} a_{n,m}(q) \frac{x^n}{[n]_q!} = \left[\sum_{n=0}^{\infty} \left(\frac{x^{mn}}{[mn]_q!} - \frac{x^{mn+1}}{[mn+1]_q!} \right) \right]^{-1}.$$

Eulerian Polynomials

Lemma (Gessel 1977)

$$\sum_L t^{\text{des}(L)+1} r_L = (1-t) \left(1 - t \sum_{n=0}^{\infty} (1-t)^n h_n \right)^{-1}$$

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- Applying Φ yields the well-known exponential generating function for the **Eulerian polynomials** $A_n(t)$ defined by

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- Applying $\hat{\Phi}$ yields the exponential generating function for **alternating Eulerian polynomials** (Chebikin 2008).
- Applying Φ_q yields the q -exponential generating function for **q -Eulerian polynomials** (Stanley 1976).

The Run Theorem

- Gessel's [run theorem](#) (1977) can be used to derive noncommutative symmetric function formulas for counting permutations
 - ① with restrictions on increasing runs;
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 - ① with restrictions on increasing runs;
 - ② by permutation statistics that are expressible in terms of increasing runs.
- See the following papers for more applications of this method to permutation enumeration:
 - I. M. Gessel and Y. Zhuang. [Counting permutations by alternating descents](#). Electron. J. Combin., 21(4): Paper P4.23, 21 pp., 2014.
 - Y. Zhuang. [Counting permutations by runs](#). J. Combin. Theory Ser. A, 142: 147–176, 2016.
 - Y. Zhuang. [Eulerian polynomials and descent statistics](#). Adv. in Appl. Math., 90: 86–144, 2017.

These include rederivations of known results by Carlitz, Elizalde and Noy, Entringer, Petersen, Remmel, Stanley, and Stembridge.

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Homomorphisms for Inverse Statistics

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- Given a permutation statistic st , define the inverse statistic ist by $ist(\pi) = st(\pi^{-1})$ for all π .

Example

Given $\pi = 24135$ we have $\pi^{-1} = 31425$, so $ides(\pi) = 2$.

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Theorem (Z. 2018+)

*If st is a **shuffle-compatible permutation statistic**, then there is a homomorphism Φ_{ist} on \mathbf{Sym} that can be used to count permutations by the inverse statistic ist .*

The Homomorphism Φ_{idcs}

- The **Hadamard product** $*$ on formal power series in t is given by

$$\left(\sum_{n=0}^{\infty} a_n t^n \right) * \left(\sum_{n=0}^{\infty} b_n t^n \right) = \sum_{n=0}^{\infty} a_n b_n t^n.$$

- Let $\mathbb{Q}[[t*, x]]$ denote the \mathbb{Q} -algebra of formal power series in t and x , where the multiplication is Hadamard product in t .

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- Let $\mathbb{Q}[[t*, x]]$ denote the \mathbb{Q} -algebra of formal power series in t and x , where the multiplication is Hadamard product in t .
- Define $\Phi_{\text{ides}}: \mathbf{Sym} \rightarrow \mathbb{Q}[[t*, x]]$ by

$$\Phi_{\text{ides}}(\mathbf{h}_n) = \frac{t}{(1-t)^{n+1}} x^n.$$

Then for $L \vDash n$,

$$\Phi_{\text{ides}}(\mathbf{r}_L) = \sum_{\text{Comp}(\pi)=L} \frac{t^{\text{ides}(\pi)+1}}{(1-t)^{n+1}} x^n.$$

(This works because des is **shuffle-compatible**.)

Counting $\underline{12 \cdots m}$ -Avoiders by ides

- Applying Φ_{ides} to $\sum_{\substack{L \\ \text{all parts} < m}} \mathbf{r}_L = \left[\sum_{n=0}^{\infty} (\mathbf{h}_{mn} - \mathbf{h}_{mn+1}) \right]^{-1}$ yields:

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Theorem (Z. 2018+)

Let $M_{m,n}^{\text{ides}}(t) := \sum_{\pi \in \text{Av}_n(\underline{12 \cdots m})} t^{\text{ides}(\pi)+1}$. For every $n \geq 1$ and $m \geq 2$, we have

$$\sum_{n=0}^{\infty} \frac{M_{m,n}^{\text{ides}}(t)}{(1-t)^{n+1}} x^n = \sum_{n=0}^{\infty} \left[\frac{t}{(1-t)^2} x - \sum_{k=1}^{\infty} \left(\frac{t}{(1-t)^{mk+1}} x^{mk} - \frac{t}{(1-t)^{mk+2}} x^{mk+1} \right) \right]^{*n}.$$

- Problem:** This is not a “Hadamard product-free” formula!

Counting $\underline{12 \cdots m}$ -Avoiders by ides—Take Two!

Theorem (Z. 2018+)

Let $\omega := e^{2\pi i/m}$. For every $n \geq 1$ and $m \geq 2$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{M_{m,n}^{\text{ides}}(t)}{(1-t)^{n+1}} x^n &= m \sum_{k=0}^{\infty} \left[\sum_{j=1}^{m-1} \frac{1 - \omega^{-j}}{(1 - \omega^j x)^k} \right]^{-1} t^k \\ &= \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} \left(\binom{k+jm-1}{k-1} x^{jm} - \binom{k+jm}{k-1} x^{jm+1} \right) \right]^{-1} t^k \quad (\heartsuit) \end{aligned}$$

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- Taking the limit of (\heartsuit) as $m \rightarrow \infty$ and extracting coefficients of x^n recovers the classical identity $\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k=0}^{\infty} k^n t^k$.

Further Directions of Research

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Count $12 \cdots m$ -avoiders by **inverses of other shuffle-compatible permutation statistics**: inverse peak number, inverse left peak number, etc. Do these yield nice **Hadamard product-free** formulas?

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THANK YOU!