Pattern-avoiding permutations and Dyson Brownian motion

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Joint work with Christopher Hoffman & Douglas Rizzolo

Permutation Patterns 2018, Dartmouth College

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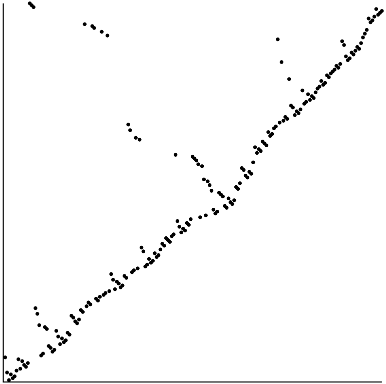


Figure: Example of **231**-avoiding permutation

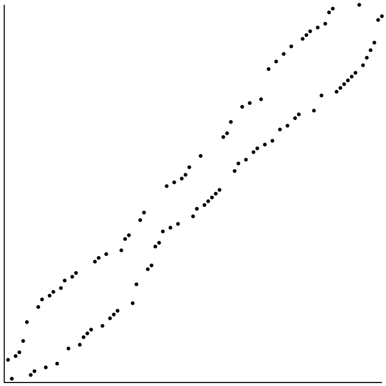


Figure: A **321**-avoiding permutation

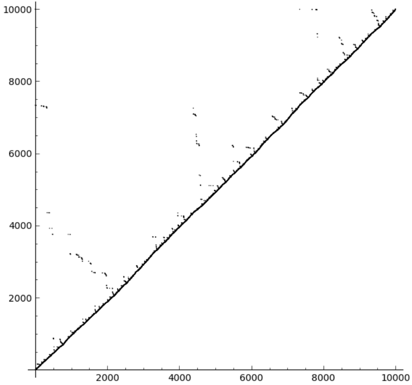


Figure: $\sigma \in S_{10000}(231)$

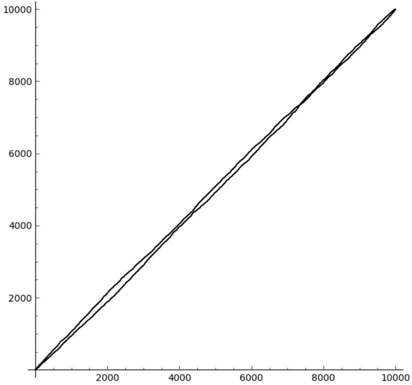


Figure: $\tau \in S_{10000}(321)$

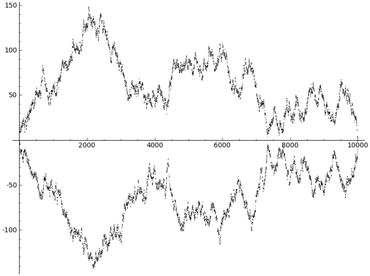


Figure: The exceedance process, $\tau(i) - i$

Some Recent Probabilistic Results

- Miner, Pak 2013 – *The shape of random pattern avoiding permutations*
- Richard Kenyon, Daniel Kral, Charles Radin, Peter Winkler 2015 – *Permutations with fixed pattern densities*
- Madras, Pehlivan 2016 – *Large deviations for permutations avoiding monotone Patterns*
- Frédérique Bassino, Mathilde Bouvel, Valentin Féray, Lucas Gerin, Adeline Pierrot, 2017 – *The Brownian limit of separable permutations*
- Janson 2018 – *Patterns in random permutations avoiding the pattern 321.*

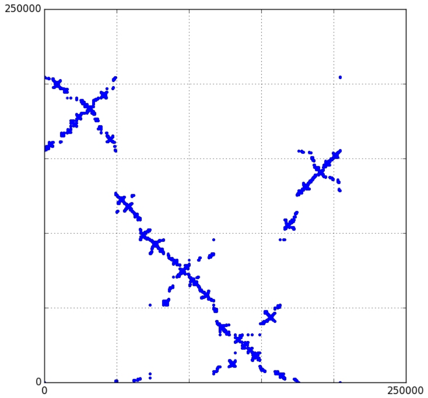


Figure: An instance of a separable permutation (from Bassino et al 2017)

Theorem (Hoffman, Rizzolo, S. '14)

Let $\Gamma^n \in \mathbf{Dyck}^{2n}$ be chosen uniformly at random. Let τ be the corresponding **321***-avoiding permutation given by the Billey-Jockusch-Stanley bijection. Let $Z(nt) := |\tau(\lfloor nt \rfloor) - nt|$. We have joint convergence of the processes*

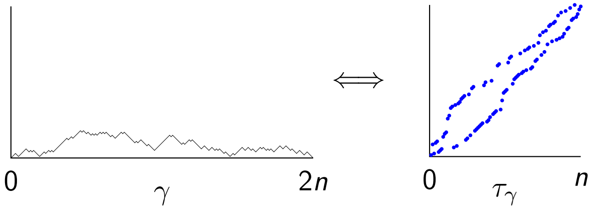
$$\frac{1}{\sqrt{2n}} Z(nt) \to \mathbbm{e}_t$$

and

$$\frac{1}{\sqrt{2n}} \Gamma^n(2ns) \to \mathbbm{e}_s$$

for all $s, t \in [0, 1]$.

A Dyck path γ and corresponding $\tau_\gamma \in S_n(321)$



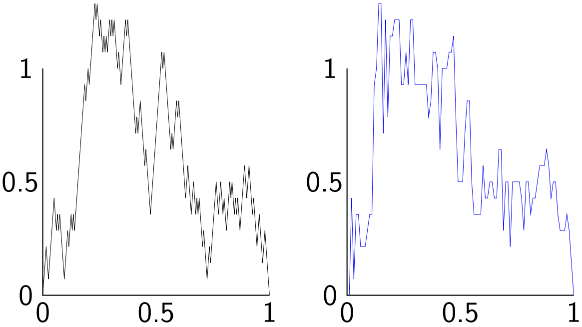
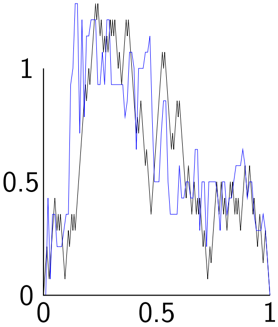


Figure: $\gamma(2nt)/\sqrt{2n}$, $Z(nt)/\sqrt{2n}$.



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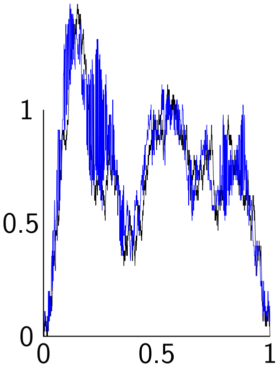

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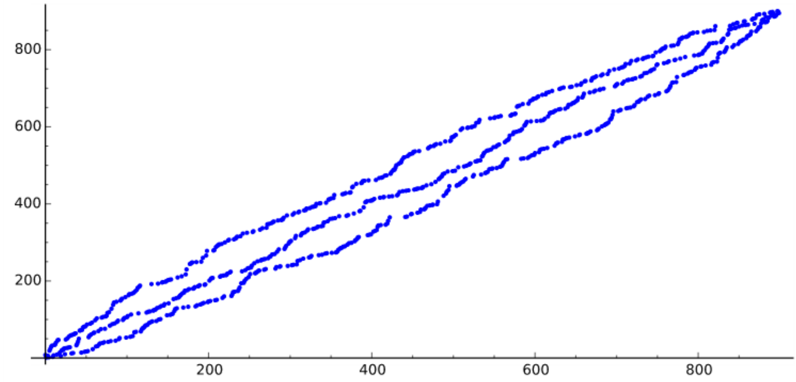

Figure: A 4321-avoiding permutation

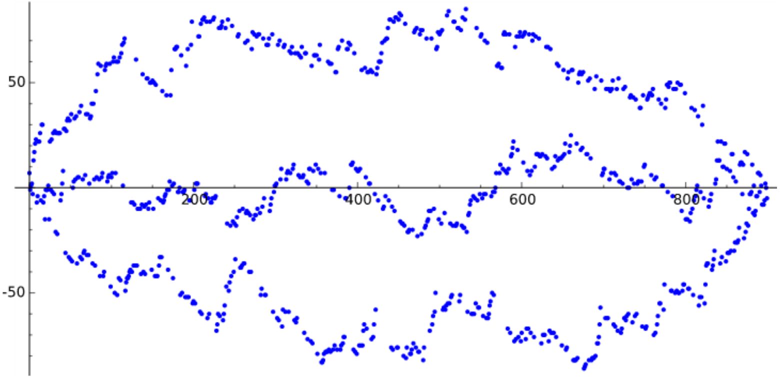

Figure: The corresponding exceedance process.

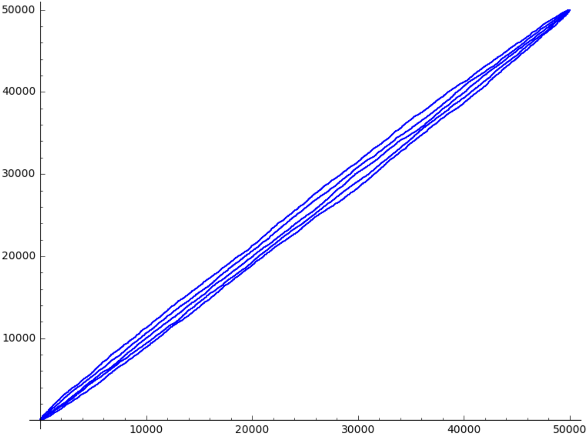

Figure: A 654321-avoiding permutation

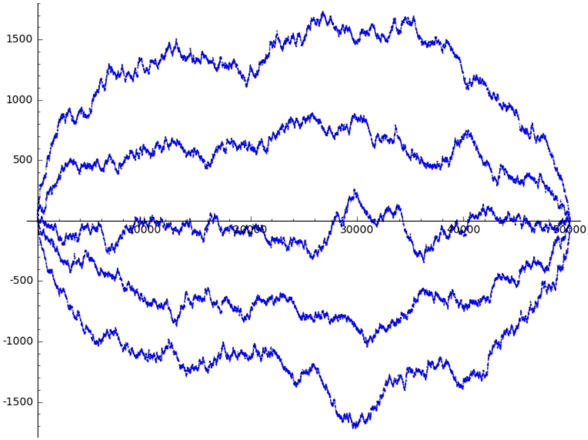

Figure: The corresponding exceedance process.

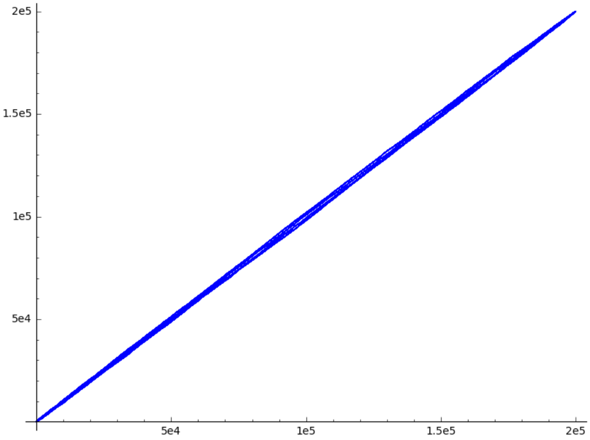

Figure: A 54321-avoiding permutation

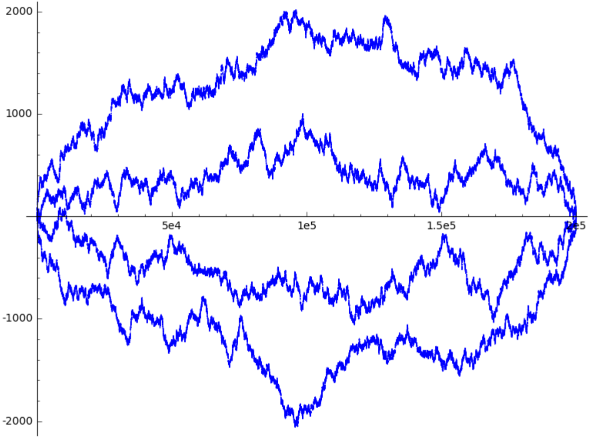

Figure: The corresponding exceedance process.

Injective map $\rho : S_n(k+1,\dots,1) \to [k]^n \times [k]^n$ by projecting the ranks of points

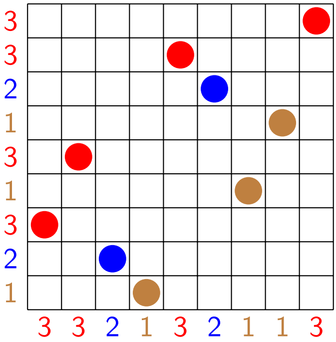

Figure: $\tau = 352182469$

- $X = (3, 3, 2, 1, 3, 2, 1, 1, 3)$ (Ranks of positions)
- $Y = (1, 2, 3, 1, 3, 1, 2, 3, 3)$ (Ranks of values)

Some notation for $\omega = (X, Y) \in [k]^n \times [k]^n$:

- $a_i^\ell :=$ # of ℓs in X before i
- $b_i^\ell :=$ # of ℓs in Y before i
- $c_i^\ell := a_i^\ell - b_i^\ell$
- $u_s^\ell :=$ location of sth ℓ in X
- $v_s^\ell :=$ location of sth ℓ in Y
- $\Omega_n : \{\omega \in [k]^n \times [k]^n : c_n^\ell = 0$ for all $\ell \in [k]\}$.
- $\rho : S_n \to \Omega_n$ is injective.

Definition

An map $\gamma : [k]^n \times [k]^n \to \mathbb{Z}^k$

$$\gamma((X,Y)) = \{(c_i^1, \cdots, c_i^k) | i \in [n]\}.$$

Some observations:

- $\sum_\ell a_i^\ell = \sum_\ell b_i^\ell = i$
- $\sum_\ell c_i^\ell = 0$
- If $S = \gamma((X,Y))$ then $S(t+1) - S(t) = e_i - e_j$ for some i and j.
- If $\omega \in \Omega_n$ then $S_\omega(n) = 0$.
- $\gamma \circ \rho : S_n \to \Omega_n$ is injective.

Definition (Traceless Dyson Brownian Motion)

Let $\lambda_1, \cdots, \lambda_k$ be Brownian bridges on $[0,1]$ conditioned to satisfy

- $\lambda_1(t) \leq \cdots \leq \lambda_k(t)$
- $\sum_{i=1}^{k} \lambda_i(t) = 0$

for all $t \in [0,1]$. We define the traceless Dyson Brownian motion as

$$\Lambda(t) = (\lambda_1(t), \cdots, \lambda_k(t)).$$

Lemma

$$\left(\frac{1}{\sqrt{n}} S_\omega | \omega \in B_n \right) \longrightarrow_d \Lambda$$

Theorem (Hoffman, Rizzolo, S. 2018)

For a permutation $\sigma \in S_n(k+1\cdots 1)$ *and* $1 \leq \ell \leq k$ *let* (u_i^ℓ, v_i^ℓ) *be the* i*th point of rank* ℓ *in* σ. *Define the scaled set of points*

$$s_\sigma^\ell(i) = \left(\frac{u_i^\ell}{n+1}, \frac{v_i^\ell - u_i^\ell}{\sqrt{n}}\right)$$

and let $\hat{s}_\sigma^\ell$ *be the linear interpolation of* s_σ^ℓ *and the points* $(0,0)$, $(1,0)$. *Finally let* Λ *be a traceless Dyson Brownian motion. Then,*

$$(\hat{s}_\sigma^1, \cdots, \hat{s}_\sigma^k) \to_d \Lambda.$$

Idea of Proof:

Use Simple Random Walk in $\gamma([k]^n \times [k]^n)$ conditioned to start and end at 0 and remain in the cone

$$\text{Cone} := \{(z_1, \cdots, z_k) : z_1 \leq \cdots \leq z_k\}$$

Read *Random Walks in Cones* (Denisov and Wachtel 2015). Use there estimates to show a random walk in $\gamma(\Omega_n) \cap \text{Cone}$ converges to Traceless Dyson Brownian Motion.

Problem: $\gamma \circ \rho(\sigma)$ is not always in Cone...

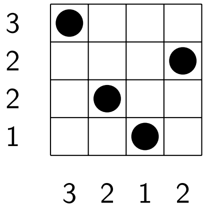

Figure: $S = \gamma \circ \rho(4213)$ and $S(3) = (0, -1, 1) \notin$ Cone.

These issues typically occur at the beginning and the end of the permutation and rare in the middle.

- Most walks that do not get too far outside the cone stay away from the boundary for the bulk of the walk.
- Most permutations have walks that spend most of the time well inside the interior of the cone.
- Create a coupling between uniform measure on walks in cones that start and end at 0 and uniform measure on walks of permutations avoiding a monotone decreasing sequence of length $k+1$.
- Show that the coupled walks are close for most of the time.

Thanks!