Stack Sorting Tiers

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July 13, 2018

Permutation Patterns 2018
Dartmouth College

Table of contents

Outline

- We consider iterating a basic stack sorting algorithm for permutations and show we get natural permutation classes arising with explicit bases.
- We then show a bijection with another class of sequences that already has a known generating function.
- Then we develop a recurrence to get an explicit generating function these permutation classes.

Sorting Algorithm

To sort a permutation we pass through a single stack and only pop the appropriate output values. If there are values left in the stack, we repeat the algorithm on the remaining stack values.

Def. If a minimum of k-passes are required to sort a permutation α, we say α has *tier* $k-1$, write $t(\alpha) = k-1$.

Examples

- The permutation 231 requires 2-passes through the stack, hence has tier 1. All other length 3 permutations are 1-pass sortable thus have tier 0.
- The permutation 231564 requires 3-passes to sort and has tier 2.
- The permutation 6427135 has tier 3.
- 4536241 has tier 4

Basics

- **Thm.** The set of permutations that requires at most k-passes (or tier at most $k-1$) to sort forms a permutation class. Denote the basis as B_k. (i.e. $t(\alpha) \leq k-1$ iff $\alpha \in Av(B_k)$)
- **Lem.** The tier of a permutation is at least as large as any permutation contained in it. If α is contained in β, then $t(\alpha) \leq t(\beta)$.
- **Thm.** Each B_k is a finite set.
- **Lem.** If $\beta \in B_k$ then the length of β is at most $3k$.

Bases for low tier

- **Thm.** (Knuth) $t(\alpha) = 0$ iff $\alpha \in Av(231)$.
- **Thm.** A permutation is 2-pass sortable (or $t(\alpha) \leq 1$) iff α avoids

$$24153, 24513, 24531, 34151, 35241, 42513, 42531, 45231,$$

$$261453, 231564, 523164$$

- Computationally we found the basis B_3 also, there are 4 basis elements of length 6, 116 of length 7, 67 of length 8 and 12 of length 9.

Maximal tier

- **Thm.** If $\alpha \in S_n$ then $t(\alpha) \leq n - 1 - \lfloor \log_2(n) \rfloor$ and the bound is sharp for all n.
  Thus, the maximal tier increases by one as a function of n except at powers of 2.
- **Thm.** If $n = 2^k - 1$ there is exactly one permutation of length n and maximal tier.
  Ex. When $n = 3$, then 2 3 1 has tier 1,
  When $n = 7$, then 4 6 3 7 2 5 1 has tier 4
  When $n = 15$, then 8 12 7 14 6 11 5 15 4 10 3 13 2 9 1

Computational Data

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6
n = 1	1						
n = 2	2						
n = 3	5	1					
n = 4	14	10					
n = 5	42	70	8				
n = 6	132	424	160	4			
n = 7	429	2382	1978	250	1		
n = 8	1430	12804	19508	6276	302		
n = 9	4862	66946	168608	106492	15674	298	
n = 10	16796	343772	1337684	1445208	451948	33148	244

Table: Number of permutations of length n and exact tier t, OEIS A122890 and A158830

OEIS Descriptions

Both OEIS A122890 and A158830 were created by manipulating generating functions, in particular-

- OEIS 158830 Construction: Let $\hat{C}(x) = xC(x)$ where $C(x)$ is the Catalan g.f. $\frac{1-\sqrt{1-4x}}{2x}$. Create an array whose n-th row is the coefficients of the n-th iterate of $\hat{C}(x)$, then multiply the g.f. for the k-th column by $(1-x)^k$.
- OEIS 122890 Construction, Let $a_1(x) = x$, $a_2(x) = x + x^2$ and for all $n \geq 2$, let $a_n(x)$ be the $n-1$ iterate of $x + x^2$. Write the coefficients of $a_n(x)$ as the entries in row n and multiply the g.f. for column j by $(1-x)^j$. (Note this has row entries reversed)

Parker's version

In Susan Parker's thesis, she gives a combinatorial description for the 2 variable generating function represented by the table.

Parker Sequences (modified) Consider sequences $a_n a_{n-1} a_{n-2} \ldots a_1$ such that for each i, $1 \leq a_{n-i+1} \leq i$. Such a sequence has a *descent* at $i+1$ if $a_{i+1} > a_i$.

Examples: $n =$ length, $t =$ number of descents.

- 12333, $n = 5, t = 0$
- 12142, $n = 5$ $t = 2$
- 1214321, $n = 7$, $t = 4$

Parker sequences cont.

- **Thm.** (Parker) The entry in the n-th row and $t+1$-st column is the number of these sequences of length n with exactly t descents.
- **Thm.** The number of Parker sequences of length n and t descents is exactly the number of permutations with length n and tier t.

Generating function new version

Let $T(n,t)$ denote the number of permutations with length n and tier t (row n column $t+1$ in table).

Let $\tau(z,w) = \sum_n \sum_k T(n,t) z^n w^t$.

Let $\rho_0(z,w) = C(z(1-w))$,

$$\rho_j(z,w) = C\left(z(1-w) \prod_{i=0}^{j-1} \rho_i(z,w) \right).$$

Thm. $\tau(z,w) = \sum_{j \geq 0} (\rho_j(z,w) - 1) w^j \prod_{i=0}^{j-1} \rho_i(z,w)$

new version cont.

Let $\psi_0(z,w) = 1 - 2z(1-w)$, $\psi_1(z,w) = \sqrt{1-4z(1-w)}$
and for $j \geq 2$, $\psi_j(z,w) = \sqrt{2\psi_{j-1}(z,w) - 1}$.
Note $\rho_j(z,w) = \frac{1-\psi_{j+1}(z,w)}{1-\psi_j(z,w)}$, $\displaystyle\prod_{i=0}^{j-1} \rho_i(z,w) = \frac{1-\psi_{j+1}(z,w)}{2z(1-w)}$.

Thm. $\displaystyle\tau(z,w) = \sum_{j\geq 0} \frac{\psi_j(z,w) - \psi_{j+1}(z,w)}{2z(1-w)} w^j$

Explicit examples

- Tier 0 (w^0) term is $\frac{1-2z-\sqrt{1-4z}}{2z} = C(z) - 1$.
- Tier 1 (w^1) term is $\frac{1-\sqrt{2\sqrt{1-4z}-1}}{2z} - \frac{1}{\sqrt{1-4z}}$
  $= z^3 + 10z^4 + 70z^5 + 424z^6 + 2382z^7 + 12804z^8 + 66946z^9 + 343772z^{10} + \ldots$
- Tier 2 (w^2) term is
  $\frac{1-\sqrt{2\sqrt{2\sqrt{1-4z}-1}-1}}{2z} + \frac{z}{\sqrt{1-4z}^3} - \frac{1}{\sqrt{1-4z}\sqrt{2\sqrt{1-4z}-1}}$
  $= 8z^5 + 160z^6 + 1978z^7 + 19508z^8 + 168608z^9 + 1337684z^{10} + 10003422z^{11} + \ldots$

Separated Pairs

Def. Given $\alpha \in S_n$ and $1 \leq i \leq n-1$, we say $(i, i+1)$ are a *(down) separated pair* if there is a subsequence in α of the form $\{(i+1)\ k\ i\}$ in α with $k > i+1$. (*A Covincular Pattern*)

- 231564 has separated pairs $(1, 2)$ and $(4, 5)$. Note $t(231564) = 2$.
- 4637251 has separated pairs $(i, i+1)$ for $i = 1, 2, 3$, and 5, note tier 4.

Thm. The tier of a permutation is exactly the number of separated pairs in it.

Parker sequences

The relation to Parker sequences is due to a bijection.
Define a function f from Parker sequences to permutations by $f(a_n a_{n-1}, \dots a_2 a_1) = \pi_n \pi_{n-1} \dots \pi_2 \pi_1$, where the element 1 is placed in position π_{a_1} and for each $j \geq 2$, place j in the a_j-th remaining position, counting from the right.

For example $f(12142)$ is given by $***1*$.
$f(12142) \to 2**1*$ (note the descent in the sequence)
$f(12142) \to 2**13$
$f(12142) \to 24*13$ (note descent)
$f(12142) \to 24513$ (note $(1,2)$ and $(3,4)$ are separated pairs)

Parker seq. cont.

Thm. The map f is a bijection from Parker sequences of length n to permutations of the same length. Further the number of descents in the sequence is equal to the number of separated pairs (and hence the tier) of the permutation.

Towards the g.f.

The notion of separated pairs also allows us to develop a recurrence on the length for permutations of a given tier. In particular,

- let $P(n, t, k)$ be the number of permutations of length n, tier t, with a 1 in the k-th position (from left),
- define $g_k : S_n \to S_{n+1}$ where $g_k(\alpha)$ results from raising all entries in α by 1 and then inserting a 1 in the k-th position.
- Ex. $g_5(23145) = 342516$ (note $(1, 2)$ sep.).

Recurrence

- Note all permutations are created by a sequence of these maps.
- For $i \geq 2$ $(i, i+1)$ sep. in $g_k(\alpha)$ iff $(i-1, i)$ sep. in α. So at least as many separated pairs in the image. Question is $(1, 2)$?
- $(1, 2)$ are sep. in $g_k(\alpha)$ iff the 1 of α is in a position $\leq k-2$.
- **Thm.** For all $n > 0, t \geq 0, 0 \leq k \leq n$

$$P(n+1, t, k) = \sum_{j \geq k-1} P(n, t, j) + \sum_{j \leq k-2} P(n, t-1, j)$$

Thanks!

Ref: Mansour, Smith, Skogman "Passing through a stack k times", on arxiv.

Thank you!

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