Classical pattern distribution in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$

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Based on joint work with Jeffrey Remmel

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In Memory of Jeffrey Remmel

Outline

1. Motivation
2. Introduction
3. Wilf-equivalence of $Q_\lambda^\gamma(t,x)$
4. Recursions of $Q_\lambda^\gamma(t,x)$
5. Other Results and Open Problems

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Motivation

Ran Pan's Project P *Project*

http://www.math.ucsd.edu/∼projectp/

Problem 13: enumerate permutations in $\mathcal{S}_n$ avoiding a classical pattern and a consecutive pattern at the same time.

Then Professor Remmel conducted researches on distribution of classical patterns and consecutive patterns in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$.

Outline

Permutations, Descents, LRmins

- A permutation $\sigma = \sigma_1 \cdots \sigma_n$ of $[n] = \{1, \ldots, n\}$ is a rearrangement of the numbers $1, \ldots, n$.
- The set of permutations of $[n]$ is denoted by $\mathcal{S}_n$.
- σ_i is a descent if $\sigma_i > \sigma_{i+1}$. $des(\sigma)$ is the number of descents in σ.
- We let $LRmin(\sigma)$ denote the number of left to right minima of σ.

Inversions, Coinversions

- (σ_i, σ_j) is an inversion if $i < j$ and $\sigma_i > \sigma_j$.
- $inv(\sigma)$ denotes the number of inversions in σ.

- (σ_i, σ_j) is a coinversion if $i < j$ and $\sigma_i < \sigma_j$.
- $coinv(\sigma)$ denotes the number of coinversions in σ.

Reduction of A Sequence

Given a sequence of distinct positive integers $w = w_1 \ldots w_n$, we let the reduction (or standardization) of the sequence, $red(w)$, denote the permutation of $[n]$ obtained from w by replacing the i-th smallest letter in w by i.

Example

If $w = 4592$, then $red(w) = 2341$.

Classical Patterns Occurrence and Avoidance

- Given a permutation $\tau = \tau_1 \dots \tau_j$ in S_j,
- we say the pattern τ occurs in $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{S}_n$ if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $red(\sigma_{i_1} \dots \sigma_{i_j}) = \tau$.
- We let $occr_\tau(\sigma)$ denote the number of τ occurrence in σ.
- We say σ avoids the pattern τ if τ does not occur in σ.

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Example

$\pi = 867932451$ avoids pattern 132, contains pattern 123. $occr_{123}(\pi) = 2$ since pattern occurrences are $6, 7, 9$ and $3, 4, 5$.

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$\pi = 867932451$ avoids pattern 132, contains pattern 123. $occr_{123}(\pi) = 2$ since pattern occurrences are $6, 7, 9$ and $3, 4, 5$.

- τ is called a classical pattern.
- inversion $\longrightarrow$ pattern 21, coinversion $\longrightarrow$ pattern 12.

$\mathcal{S}_n(\sigma)$

- We let $\mathcal{S}_n(\lambda)$ denote the set of permutations in $\mathcal{S}_n$ avoiding λ.
- $\left|\mathcal{S}_n(132)\right| = \left|\mathcal{S}_n(123)\right| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number.
- C_n is also the number of $n \times n$ Dyck paths.
- Let $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$, then $\mathcal{S}_n(\Lambda)$ is the set of permutations in $\mathcal{S}_n$ avoiding $\lambda_1, \ldots, \lambda_r$.

Our Problem

Given two sets of permutations $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ and $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$, we study the distribution of classical patterns $\gamma_1, \ldots, \gamma_s$ in $\mathcal{S}_n(\Lambda)$.

Especially, we study pattern τ distribution in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$ in the case when τ is of length 3 and some special form.

Generating Function

We define

$$Q^{\Gamma}_{\Lambda}(t, x_1, \ldots, x_s) = 1 + \sum_{n \geq 1} t^n Q^{\Gamma}_{n,\Lambda}(x_1, \ldots, x_s),$$

where

$$Q^{\Gamma}_{n,\Lambda}(x_1, \ldots, x_s) = \sum_{\sigma \in \mathcal{S}_n(\Lambda)} x_1^{occr_{\gamma_1}(\sigma)} \cdots x_s^{occr_{\gamma_s}(\sigma)}.$$

Especially, we have

$$Q^{\gamma}_{\lambda}(t, x) = 1 + \sum_{n \geq 1} t^n Q^{\gamma}_{n,\lambda}(x) \quad \text{and} \quad Q^{\gamma}_{n,\lambda}(x) = \sum_{\sigma \in \mathcal{S}_n(\lambda)} x^{occr_{\gamma}(\sigma)}.$$

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Wilf-equivalence

Given a permutation σ, we denote the reverse of σ by σ^r, the complement of σ by σ^c, the reverse-complement of σ by σ^{rc}, and the inverse of σ by σ^{-1}.

Example

Let $\sigma = 15324$, then
$\sigma^r = 42351,\ \sigma^c = 51342,\ \sigma^{rc} = 24315,\ \sigma^{-1} = 14352.$

Wilf-equivalence

- $\mathcal{S}_n(123)$ is closed under the operation reverse-complement.
- Both $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ are closed under the operation inverse.

Thus,

Theorem

Given any permutation pattern γ,

$$Q_{123}^{\gamma}(t,x) = Q_{123}^{\gamma^{rc}}(t,x) = Q_{123}^{\gamma^{-1}}(t,x), \quad Q_{132}^{\gamma}(t,x) = Q_{132}^{\gamma^{-1}}(t,x).$$

Wilf-equivalence

When we let γ be a pattern of length 3,

Corollary

There are 4 Wilf-equivalent classes for $\mathcal{S}_n(132)$,

(1) $Q_{132}^{123}(t,x)$,
(2) $Q_{132}^{213}(t,x)$,
(3) $Q_{132}^{231}(t,x) = Q_{132}^{312}(t,x)$,
(4) $Q_{132}^{321}(t,x)$,

and there are 3 Wilf-equivalent classes for $\mathcal{S}_n(123)$,

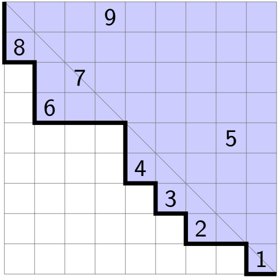
(1) $Q_{123}^{132}(t,x) = Q_{123}^{213}(t,x)$,
(2) $Q_{123}^{231}(t,x) = Q_{123}^{312}(t,x)$,
(3) $Q_{123}^{321}(t,x)$.

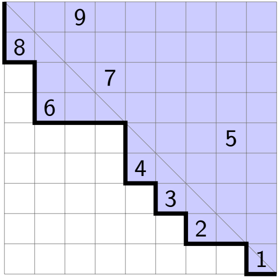
Outline

Method – Using Dyck Path Bijections

We use Dyck path bijections to calculate the recursive formulas for $Q_{\lambda}^{\gamma}(t,x)$.

Krattenthaler $\Phi : S_n(132) \to D_n$, Elizalde and Deutsch $\Psi : \mathcal{S}_n(123) \to \mathcal{D}_n$.



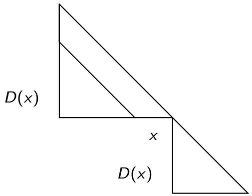


Method – Using Dyck Path Bijections

Then, we the recursion of Dyck path by breaking the path at the first place it hits the diagonal to break it into 2 Dyck paths.

Let $D(x)$ be the generating function enumerating the number of Dyck paths of size n,

$$D(x) = 1 + xD(x)^2.$$



Recursion of Dyck path

Counting Length 2 pattern in $\mathcal{S}_n(132)$

We first consider permutations that are avoiding 132 and the distribution of pattern of length 2, i.e. inv and coinv.

We let

$$Q_n(q) = Q^{12}_{n,132}(q) = \sum_{\sigma \in \mathcal{S}_n(132)} q^{coinv(\sigma)},$$

$$Q(t,q) = Q^{12}_{132}(t,q) = 1 + \sum_{n \geq 1} t^n \sum_{\sigma \in \mathcal{S}_n(132)} q^{coinv(\sigma)},$$

$$\text{and} \quad P_n(p,q) = \sum_{\sigma \in \mathcal{S}_n(132)} p^{inv(\sigma)} q^{coinv(\sigma)}.$$

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$$\text{and} \quad P_n(p,q) = \sum_{\sigma \in \mathcal{S}_n(132)} p^{inv(\sigma)} q^{coinv(\sigma)}.$$

Since $inv(\sigma) + coinv(\sigma) = \binom{n}{2}$, we have the following relation about $P_n(p,q)$ and $Q_n(q)$,

$$P_n(p,q) = \sum_{\sigma \in \mathcal{S}_n(132)} p^{\binom{n}{2} - coinv(\sigma)} q^{coinv(\sigma)} = p^{\binom{n}{2}} Q_n\left(\frac{q}{p}\right).$$

Counting Length 2 pattern in $\mathcal{S}_n(132)$

$Q_n(q)$ – q-Catalan number.

Theorem (Fürlinger and Hofbauer)

Let $Q_n(q) = Q^{12}_{n,132}(q)$ *and* $Q(t,q) = Q^{12}_{132}(t,q)$, *then we have the recursions,*

$$Q_0(q) = 1, \;\; Q_n(q) = \sum_{k=1}^{n} q^{k-1} Q_{k-1}(q) Q_{n-k}(q), \tag{1}$$

$$P_0(q) = 1, \;\; P_n(q) = \sum_{k=1}^{n} q^{k(n-k)} P_{k-1}(q) P_{n-k}(q), \tag{2}$$

and we have the functional equation,

$$Q(t,q) = 1 + tQ(t,q) \cdot Q(tq,q). \tag{3}$$

Counting Length 3 pattern in $\mathcal{S}_n(132)$

Theorem

We let $Q^{\gamma}_{n,132}(q,x) = \sum_{\sigma\in\mathcal{S}_n(132)} q^{coinv(\sigma)} x^{occr_{\gamma}(\sigma)}$, then we have the following recursive equations for the generating function $Q^{\gamma}_{n,132}(q,x)$.

$$
\begin{aligned}
Q^{\gamma}_{0,132}(q,x) &= 1 \quad \text{for each pattern } \gamma, & (4)\\
Q^{123}_{n,132}(q,x) &= \sum_{k=1}^{n} q^{k-1} Q_{k-1}(qx,x) Q_{n-k}(q,x), & (5)\\
Q^{213}_{n,132}(q,x) &= \sum_{k=1}^{n} q^{k-1} x^{\frac{(k-1)(k-2)}{2}} Q_{k-1}(\frac{q}{x},x) Q_{n-k}(q,x), & (6)\\
Q^{231}_{n,132}(q,x) &= \sum_{k=1}^{n} q^{k-1} x^{(k-1)(n-k)} Q_{k-1}(qx^{(n-k)},x) Q_{n-k}(q,x), & (7)\\
Q^{321}_{n,132}(q,x) &= \sum_{k=1}^{n} q^{k-1} x^{\frac{(n-k)(kn-4k+2)}{2}} Q_{k-1}(\frac{q}{x^{n-k}},x) Q_{n-k}(\frac{q}{x^{k}},x). & (8)
\end{aligned}
$$

Track all patterns of length 2 and 3 in $\mathcal{S}_n(132)$

We can also track all the patterns that

$$
\begin{aligned}
& Q_{n,132}^{12,21,123,213,231,312,321}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) \\
= \; & \sum_{k=1}^{n} x_1^{k-1} x_2^{k(n-k)} x_5^{(k-1)(n-k)} \\
& \quad \cdot Q_{k-1}(x_1 x_3 x_5^{(n-k)}, x_2 x_4 x_7^{(n-k)}, x_3, x_4, x_5, x_6, x_7) \\
& \quad \cdot Q_{n-k}(x_1 x_6^{k}, x_2 x_7^{k}, x_3, x_4, x_5, x_6, x_7).
\end{aligned} \tag{9}
$$

Track all patterns of length 2 and 3 in $\mathcal{S}_n(132)$

Expansion of $Q_{n,132}^{12,21,123,213,231,312,321}(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$

n	$Q_{n,132}^{12,21,123,213,231,312,321}(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$
0	1
1	1
2	x_1+x_2
3	$x_1^3x_7+x_1^2x_2x_5+x_1^2x_2x_6+x_1x_2^2x_4+x_2^3x_3$
4	$x_1^6x_7^4+x_1^5x_2x_5^2x_7^2+x_1^5x_2x_5x_6x_7^2+x_1^5x_2x_6^2x_7^2+x_1^4x_2^2x_4x_5^2x_7+x_1^4x_2^2x_4x_6^2x_7+x_1^4x_2^2x_5^2x_6^2$ $+x_1^3x_2^3x_3x_5^3+x_1^3x_2^3x_3x_6^3+x_1^3x_2^3x_4^3x_7+x_1^2x_2^4x_3x_4^2x_5+x_1^2x_2^4x_3x_4^2x_6+x_1x_2^5x_3^2x_4^2+x_2^6x_3^4$
5	$x_1^{10}x_7^{10}+x_1^9x_2x_5^3x_7^7+x_1^9x_2x_5^2x_6x_7^7+x_1^9x_2x_5x_6^2x_7^7+x_1^9x_2x_6^3x_7^7+x_1^8x_2^2x_4x_5^4x_7^5+x_1^8x_2^2x_4x_5^2x_6^2x_7^5$ $+x_1^8x_2^2x_4x_6^4x_7^5+x_1^8x_2^2x_5^4x_6^2x_7^4+x_1^8x_2^2x_5^3x_6^3x_7^4+x_1^8x_2^2x_5^2x_6^4x_7^4+x_1^7x_2^3x_3x_5^6x_7^3+x_1^7x_2^3x_3x_5^3x_6^3x_7^3$ $+x_1^7x_2^3x_3x_6^6x_7^3+x_1^7x_2^3x_4^3x_5^3x_7^4+x_1^7x_2^3x_4^3x_6^3x_7^4+x_1^7x_2^3x_4x_5^4x_6^3x_7^2+x_1^7x_2^3x_4x_5^3x_6^4x_7^2+x_1^6x_2^4x_3x_4^2x_5^5x_7^2$ $+x_1^6x_2^4x_3x_4^2x_5^4x_6x_7^2+x_1^6x_2^4x_3x_4^2x_5x_6^4x_7^2+x_1^6x_2^4x_3x_4^2x_6^5x_7^2+x_1^6x_2^4x_3x_5^6x_6^3+x_1^6x_2^4x_3x_5^3x_6^6$ $+x_1^6x_2^4x_4^6x_7^4+x_1^5x_2^5x_3^2x_4^2x_5^5x_7+x_1^5x_2^5x_3^2x_4^2x_6^5x_7+x_1^5x_2^5x_3x_4^5x_5^2x_7^2+x_1^5x_2^5x_3x_4^5x_5x_6x_7^2$ $+x_1^5x_2^5x_3x_4^5x_6^2x_7^2+x_1^4x_2^6x_3^4x_5^6+x_1^4x_2^6x_3^4x_6^6+x_1^4x_2^6x_3^2x_4^5x_5^2x_7+x_1^4x_2^6x_3^2x_4^5x_6^2x_7+x_1^4x_2^6x_3^2x_4^4x_5^2x_6^2$ $+x_1^3x_2^7x_3^4x_4^3x_5^3+x_1^3x_2^7x_3^4x_4^3x_6^3+x_1^3x_2^7x_3^3x_4^6x_7+x_1^2x_2^8x_3^5x_4^4x_5+x_1^2x_2^8x_3^5x_4^4x_6+x_1x_2^9x_3^7x_4^3+x_2^{10}x_3^{10}$

Counting Length 3 pattern in $\mathcal{S}_n(132)$

We also get nice recursions for pattern distributions in $\mathcal{S}_n(123)$. For example, we have

Theorem

Let $Q_{n,123}^{132}(s,q,x) = \sum_{\sigma \in \mathcal{S}_n(123)} s^{LRmin(\sigma)} q^{coinv(\sigma)} x^{occr_{132}(\sigma)}$, *then we have the following recursions,*

$$
\begin{aligned}
Q_{0,123}^{132}(s,q,x) &= 1, \\
Q_{n,123}^{132}(s,q,x) &= sQ_{n-1} + \sum_{k=2}^{n} Q_{k-1}(sq,qx,x) Q_{n-k}(s,q,x).
\end{aligned}
$$

An equality between $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$

We get nice recursions and functional equations for the function counting pattern $12\cdots m$ in $\mathcal{S}_n(132)$ and the function counting pattern $1m(m-1)\cdots 2$ in $\mathcal{S}_n(123)$, for any $m > 1$.

We found a big coincidence among $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$ that,

$$|\{\sigma \in \mathcal{S}_n(132) : occr_{12\cdots j}(\sigma) = i\}| = |\{\sigma \in \mathcal{S}_n(123) : occr_{1j(j-1)\cdots 2}(\sigma) = i\}|,$$

for all $i < j$.

An equality between $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$

This result is described in the following theorem.

Theorem

We let

$$Q_{n,132}(x_2, x_3, \ldots, x_m) = \sum_{\sigma \in \mathcal{S}_n(132)} x_2^{occr_{12}} x_3^{occr_{123}} \cdots x_m^{occr_{12\cdots m}},$$

$$Q_{132}(t, x_2, x_3, \ldots, x_m) = \sum_{n \geq 0} t^n Q_{n,132}(x_2, x_3, \ldots, x_m) \quad and$$

$$Q_{n,123}(s, x_2, x_3, \ldots, x_m) = \sum_{\sigma \in \mathcal{S}_n(123)} s^{LRmin} x_2^{occr_{12}} x_3^{occr_{132}} \cdots x_m^{occr_{1m(m-1)\cdots 2}},$$

$$Q_{123}(t, s, x_2, x_3, \ldots, x_m) = \sum_{n \geq 0} t^n Q_{n,123}(s, x_2, x_3, \ldots, x_m),$$

Theorem

then we have the following equations,

$$Q_{n,132}(x_2, \ldots, x_m) = \sum_{k=1}^{n} x_2^{k-1} Q_{k-1,132}(x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) Q_{n-k,132}(x_2, \ldots, x_m),$$

$$Q_{n,123}(s, x_2, \ldots, x_m) = sQ_{n-1,123}(t, s, x_2, \ldots, x_m) + \sum_{k=2}^{n} Q_{k-1,123}(sx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) Q_{n-k,123}(s, x_2, \ldots, x_m),$$

Theorem

also the functional equations,

$$Q_{132}(t, x_2, \ldots, x_m)$$
$$= 1 + Q_{132}(tx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m)Q_{132}(t, x_2, \ldots, x_m),$$

$$Q_{123}(t, s, x_2, \ldots, x_m) = 1 + t(s-1)Q_{123}(t, s, x_2, \ldots, x_m)$$
$$+tQ_{123}(t, sx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m)Q_{123}(s, x_2, \ldots, x_m).$$

Further, let $[x^i]_Q$ denote the coefficient of x^i in function Q, then

$$[t^n x_i^j]_{Q_{132}} = [t^n x_j^i]_{Q_{123}} \quad \text{for} \quad i < j. \tag{10}$$

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Other Results and Open Problems

- We obtained the recursion tracking all patterns of length≤ 4 on $\mathcal{S}_n(132)$, see that every pattern is trackable on $\mathcal{S}_n(132)$.

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- There are other equality of coefficients of generating functions Q_{132}^{γ} and Q_{123}^{γ} except equation (10) which we can study in the future.
- We only studied classical patterns on $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$, and circular patterns on 1243.

Thank You!