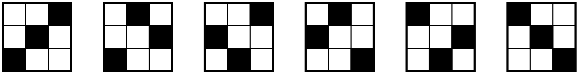
On the Growth of Merges and Staircases of Permutation Classes

Jay Pantone

Dartmouth College

Hanover, NH

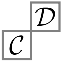
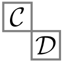


Joint work with Michael Albert and Vince Vatter.

Permutation Patterns 2018

July 10, 2018

MIXING TWO CLASSES

- Sum / skew sum  

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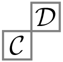

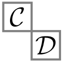

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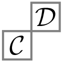
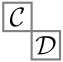
- Sum / skew sum 



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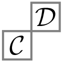
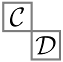
- Sum / skew sum 



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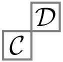

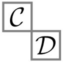

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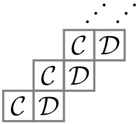

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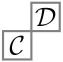

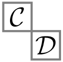

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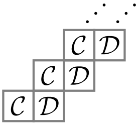
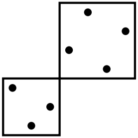


- Merge $\mathcal{C} \odot \mathcal{D}$

Sum and Skew Sum

$312 \oplus 2413 =$  $= 3125746$

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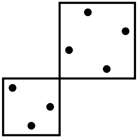
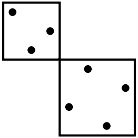
$312 \oplus 2413 =$  $= 3125746$

$312 \ominus 2413 =$  $= 7562413$

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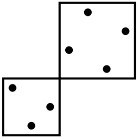
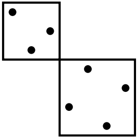
$312 \oplus 2413 =$  $= 3125746$

$312 \ominus 2413 =$  $= 7562413$

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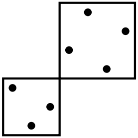
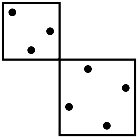
$312 \oplus 2413 =$  $= 3125746$

$312 \ominus 2413 =$  $= 7562413$

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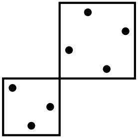
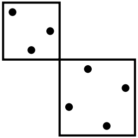
$312 \oplus 2413 =$  $= 3125746$

$312 \ominus 2413 =$  $= 7562413$

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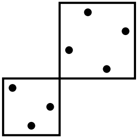
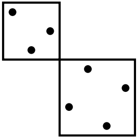
$312 \oplus 2413 =$  $= 3125746$

$312 \ominus 2413 =$  $= 7562413$

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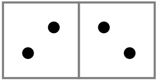
Growth rate: $\max\{\mathrm{gr}(C), \mathrm{gr}(D)\}$
Basis: tricky

JUXTAPOSITION

$12|21 =$  $= \{$  $\}$

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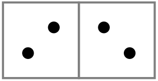
12|21 =  = {  }

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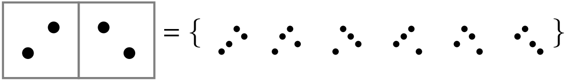

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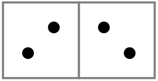
12|21 =  = {  }

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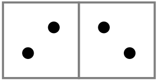
12|21 =  = {  }

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Basis: easy

Inflation

$$3142[231, 21, 123, 1]$$

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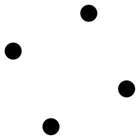

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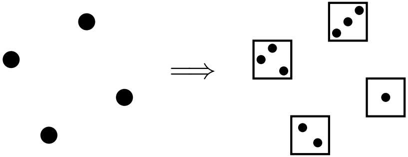

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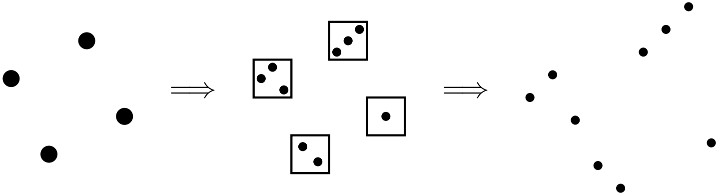

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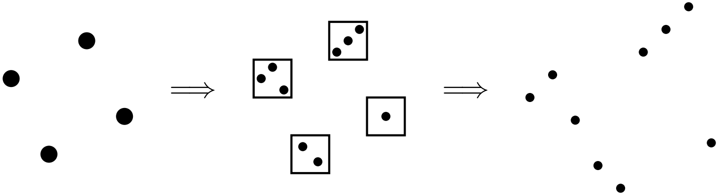

$$3142[231, 21, 123, 1] = 564217893$$

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Enumeration: 😩

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Basis: 😩

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Enumeration: 😭
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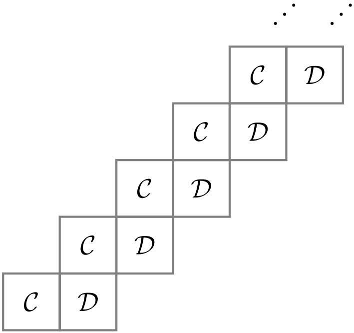
Basis: 😭

STAIRCASES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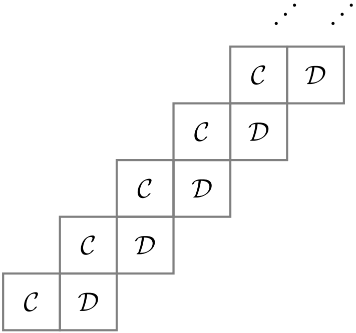
Staircases



Enumeration: 😫

Growth rate: 😫

Basis: 😫

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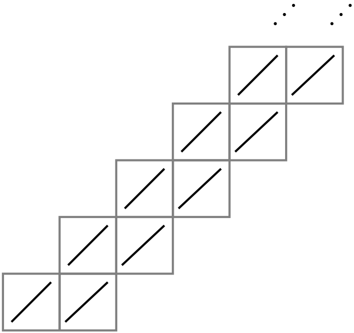
$\mathrm{Av}(321) =$ 

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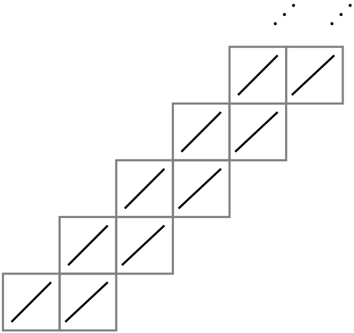

Enumeration: easy
Growth rate: easy
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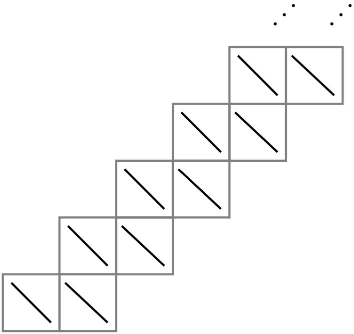

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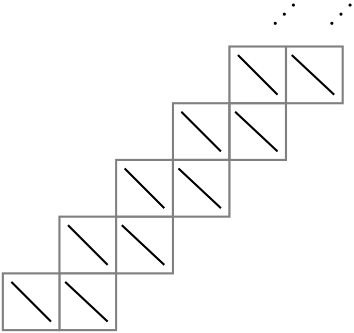

Enumeration: non-D-finite?
Growth Rate: ≈ 4.5189296247758?
Basis: infinitely based

MERGE

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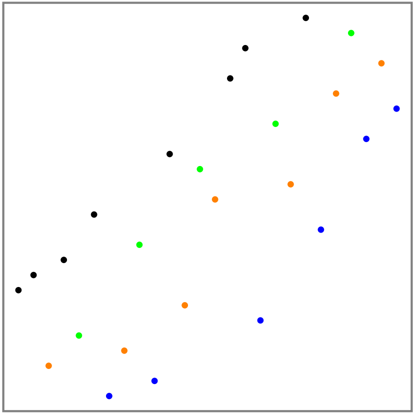
MERGE

Examples:

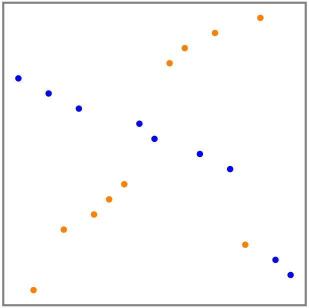
- $\operatorname{Av}(k\cdots 21) = \underbrace{\operatorname{Av}(21) \odot \cdots \odot \operatorname{Av}(21)}_{k-1 \text{ copies of } \operatorname{Av}(21)} = \operatorname{Av}((k{-}1)\cdots 21) \odot \operatorname{Av}(21)$



Merge

Examples:

- Skew-merged: $\mathrm{Av}(12) \odot \mathrm{Av}(21) = \mathrm{Av}(2143, 3412)$



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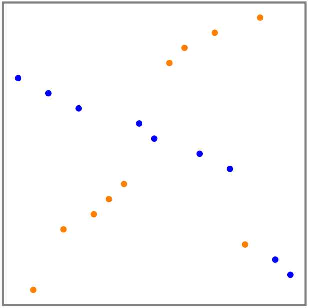

Enumeration: 🤢

Growth rate: 🤢

Basis: 🤢

STAIRCASES + MERGE

Surprisingly, there's a

lower bound on the growth rate of the $(\mathcal{C}, \mathcal{D})$-staircase, and an

upper bound on the growth rate of $\mathcal{C} \odot \mathcal{D}$

that sometimes meet in the middle in a useful way.

MERGE UPPER BOUND

Theorem. (Claesson, Jelínek, Steingrímsson)
Suppose $\mathcal{C}$ and $\mathcal{D}$ have growth rates. Then, if $\mathrm{gr}(\mathcal{C} \odot \mathcal{D})$ exists,

$$\mathrm{gr}(\mathcal{C} \odot \mathcal{D}) \leq \left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})}\right)^2.$$

Idea:

$$(\mathcal{C} \odot \mathcal{D})_n \leq \sum_{k=0}^{n} \binom{n}{k}^2 \mathcal{C}_k \mathcal{D}_{n-k}.$$

Merge Upper Bound

Application:

$$\mathrm{Av}(1324) \subseteq \mathrm{Av}(132) \odot \mathrm{Av}(213)$$

and so

$$\mathrm{gr}(\mathrm{Av}(1324)) \leq \mathrm{gr}(\mathrm{Av}(132) \odot \mathrm{Av}(213)) \leq \left(\sqrt{4} + \sqrt{4}\right)^2 = 16.$$

STAIRCASE LOWER BOUND

Given a matrix M whose entries are permutation classes, define $\mathrm{Grid}(M)$ to be the class of permutations that can be partitioned into cells such that each subpermutation is in the corresponding class.

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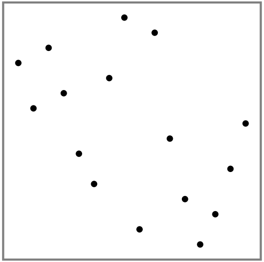

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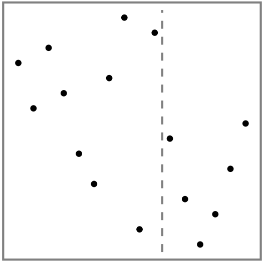

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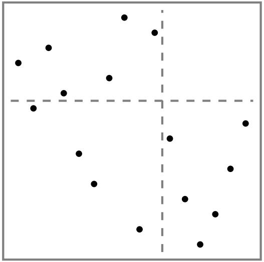

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Staircase Lower Bound

Theorem. (Albert and Vatter)
Let $\mathcal{M}$ be a $t \times u$ matrix of permutation classes, each with a proper growth rate, and define the $t \times u$ matrix Γ by $\Gamma_{k,\ell} = \sqrt{\mathrm{gr}(\mathcal{M}_{k,\ell})}$. The growth rate of $\mathrm{Grid}(\mathcal{M})$ is equal to the greatest eigenvalue of $\Gamma\Gamma^T$.

Staircase Lower Bound

Theorem. (Albert and Vatter)
Let $\mathcal{M}$ be a $t \times u$ matrix of permutation classes, each with a proper growth rate, and define the $t \times u$ matrix Γ by $\Gamma_{k,\ell} = \sqrt{\mathrm{gr}(\mathcal{M}_{k,\ell})}$. The growth rate of $\mathrm{Grid}(\mathcal{M})$ is equal to the greatest eigenvalue of $\Gamma\Gamma^T$.

$$\mathrm{gr}\left[\mathrm{Grid}\begin{pmatrix} \mathrm{Av}(321) & \emptyset \\ \mathrm{Av}(21) & \mathrm{Av}(132, 231) \end{pmatrix}\right]$$

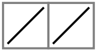
$$= \text{largest eigenvalue of } \begin{pmatrix} \sqrt{4} & 0 \\ \sqrt{1} & \sqrt{2} \end{pmatrix}\begin{pmatrix} \sqrt{4} & 0 \\ \sqrt{1} & \sqrt{2} \end{pmatrix}^T$$

$$= \frac{7+\sqrt{17}}{2} \approx 5.562.$$

STAIRCASE LOWER BOUND

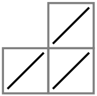
 $\longrightarrow \quad \begin{pmatrix} 1 \end{pmatrix} \quad \longrightarrow \quad \mathrm{gr}(\mathcal{C}) = 1$

STAIRCASE LOWER BOUND



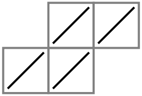
$$\longrightarrow \quad \begin{pmatrix} 1 & 1 \end{pmatrix} \quad \longrightarrow \quad \mathrm{gr}(\mathcal{C}) = 2$$

Staircase Lower Bound



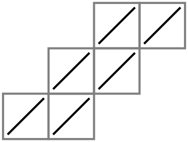
$$\longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow \operatorname{gr}(\mathcal{C}) = 1 + \phi \approx 2.618$$

Staircase Lower Bound



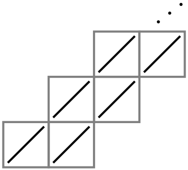
$$\longrightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \longrightarrow \mathrm{gr}(\mathcal{C}) = 3$$

Staircase Lower Bound



$$\longrightarrow \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \quad \mathrm{gr}(\mathcal{C}) \approx 3.414$$

Staircase Lower Bound



$$\longrightarrow \begin{pmatrix} 0 & 0 & 0 & \iddots \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \quad \mathrm{gr}(\mathcal{C}) \to 4.$$

STAIRCASE LOWER BOUND

Let $\text{gr}(\mathcal{C}) = \alpha$ and $\text{gr}(\mathcal{D}) = \beta$.

$$\boxed{\mathcal{C}} \quad \longrightarrow \quad (\ \sqrt{\alpha}\ ) \quad \longrightarrow \quad \text{growth rate} = \alpha$$

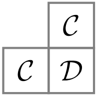
STAIRCASE LOWER BOUND

Let $\operatorname{gr}(\mathcal{C}) = \alpha$ and $\operatorname{gr}(\mathcal{D}) = \beta$.

$$\begin{array}{|c|c|}\hline \mathcal{C} & \mathcal{D} \\ \hline \end{array} \quad \longrightarrow \quad \begin{pmatrix} \sqrt{\alpha} & \sqrt{\beta} \end{pmatrix} \quad \longrightarrow \quad \text{growth rate} = \alpha + \beta$$

STAIRCASE LOWER BOUND

Let $\mathrm{gr}(\mathcal{C}) = \alpha$ and $\mathrm{gr}(\mathcal{D}) = \beta$.

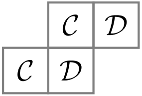


$$\longrightarrow \begin{pmatrix} 0 & \sqrt{\alpha} \\ \sqrt{\alpha} & \sqrt{\beta} \end{pmatrix} \longrightarrow$$

$$\text{growth rate} = \alpha + \frac{\beta + \sqrt{4\alpha\beta + \beta^2}}{2}$$

Staircase Lower Bound

Let $\mathrm{gr}(\mathcal{C}) = \alpha$ and $\mathrm{gr}(\mathcal{D}) = \beta$.

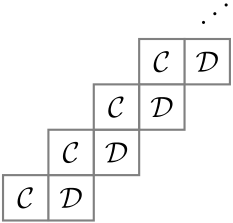


$$\longrightarrow \begin{pmatrix} 0 & \sqrt{\alpha} & \sqrt{\beta} \\ \sqrt{\alpha} & \sqrt{\beta} & 0 \end{pmatrix} \longrightarrow$$

$$\text{growth rate} = \alpha + \beta + \sqrt{\alpha\beta}$$

Staircase Lower Bound

Let $\mathrm{gr}(\mathcal{C}) = \alpha$ and $\mathrm{gr}(\mathcal{D}) = \beta$.



$$\longrightarrow \quad \Gamma = \begin{pmatrix} & & & & \iddots \\ 0 & 0 & 0 & \sqrt{\alpha} & \sqrt{\beta} \\ 0 & 0 & \sqrt{\alpha} & \sqrt{\beta} & 0 \\ 0 & \sqrt{\alpha} & \sqrt{\beta} & 0 & 0 \\ \sqrt{\alpha} & \sqrt{\beta} & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \quad \Gamma\Gamma^T = \begin{pmatrix} \alpha+\beta & \sqrt{\alpha\beta} & 0 & 0 \\ \sqrt{\alpha\beta} & \alpha+\beta & \sqrt{\alpha\beta} & 0 \\ 0 & \sqrt{\alpha\beta} & \alpha+\beta & \sqrt{\alpha\beta} \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

Staircase Lower Bound

The largest eigenvalue of $\Gamma\Gamma^T$ of this form with t rows is

$$\alpha + \beta + 2\sqrt{\alpha\beta}\cos\left(\frac{1}{t+1}\right) \to \alpha + \beta + 2\sqrt{\alpha\beta}$$
$$= \left(\sqrt{\alpha} + \sqrt{\beta}\right)^2.$$

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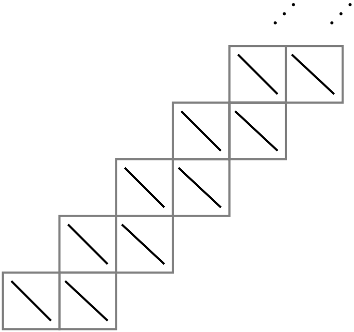
$$\alpha+\beta+2\sqrt{\alpha\beta}\cos\left(\frac{1}{t+1}\right) \to \alpha+\beta+2\sqrt{\alpha\beta}$$
$$= \left(\sqrt{\alpha}+\sqrt{\beta}\right)^2.$$

Thus,

$$\mathrm{gr}(\ (\mathcal{C},\mathcal{D})\text{-staircase}\ ) \geq \left(\sqrt{\mathrm{gr}(\mathcal{C})}+\sqrt{\mathrm{gr}(\mathcal{D})}\right)^2.$$

Staircase Lower Bound

Warning! The inequality sign is important.



Every finite truncation has growth rate < 4, but the infinite staircase has growth rate ≈ 4.5.

UPPER AND LOWER BOUNDS

So, if $\mathcal{C}$ and $\mathcal{D}$ have proper growth rates:

UPPER AND LOWER BOUNDS

So, if $\mathcal{C}$ and $\mathcal{D}$ have proper growth rates:

$$\mathrm{gr}(\mathcal{C} \odot \mathcal{D}) \leq \left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})}\right)^2$$

Upper and Lower Bounds

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UPPER AND LOWER BOUNDS

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$$\left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})}\right)^2 \leq \mathrm{gr}(\ (\mathcal{C}, \mathcal{D})\text{-staircase}\ )$$

In cases where the $(\mathcal{C}, \mathcal{D})$-staircase is contained in $\mathcal{C} \odot \mathcal{D}$, we have equality.

Main Theorem

$\mathcal{C}$ is *sum-closed* if $\pi \oplus \sigma \in \mathcal{C}$ whenever $\pi, \sigma \in \mathcal{C}$ and $\mathcal{C}$ is *skew-closed* if $\pi \ominus \sigma \in \mathcal{C}$ whenever $\pi, \sigma \in \mathcal{C}$.

MAIN THEOREM

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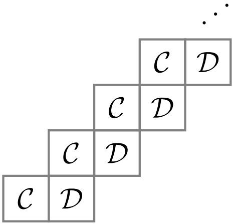
Theorem. (Albert, P., Vatter)
If each of the classes $\mathcal{C}$ and $\mathcal{D}$ is sum or skew closed, then

$$\mathrm{gr}(\mathcal{C} \odot \mathcal{D}) = \left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})}\right)^2.$$

Proof of Main Theorem

If $\mathcal{C}$ and $\mathcal{D}$ are sum-closed:



$\subseteq \mathcal{C} \odot \mathcal{D}$

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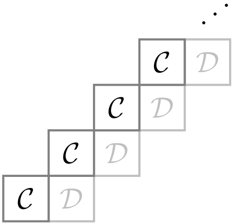

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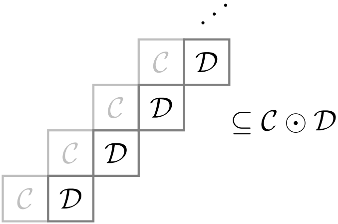

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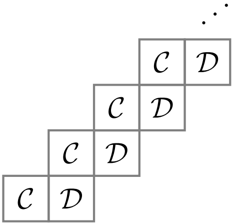

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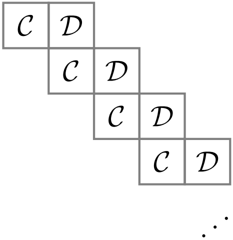

$\subseteq \mathcal{C} \odot \mathcal{D}$

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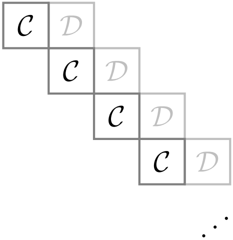

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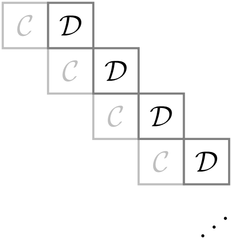

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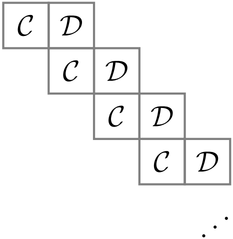

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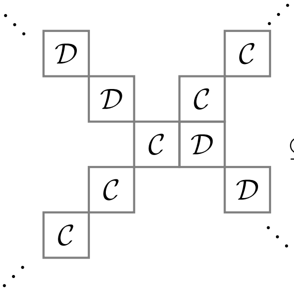
Proof of Main Theorem

If $\mathcal{C}$ is sum-closed and $\mathcal{D}$ is skew-closed:



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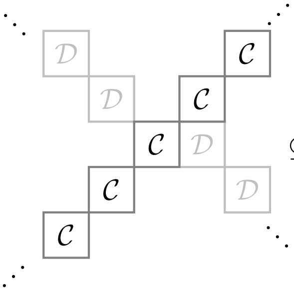

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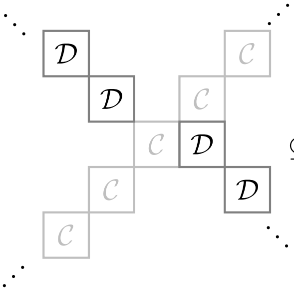

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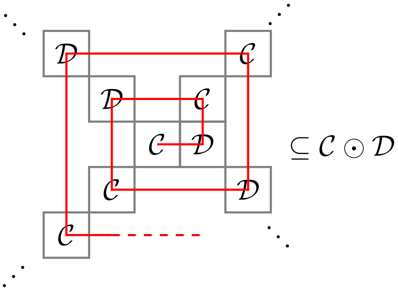

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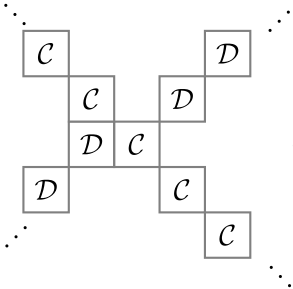

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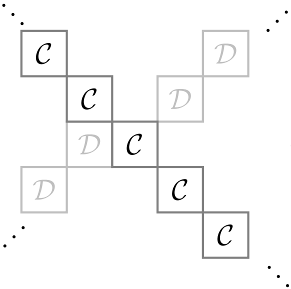

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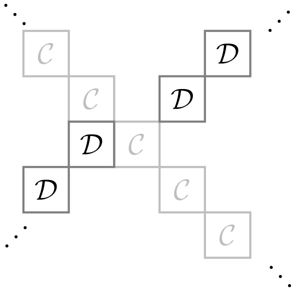

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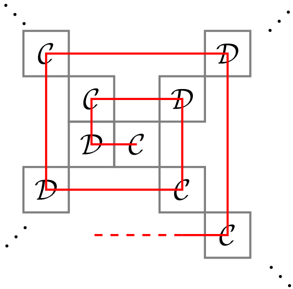

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APPLICATION #1 — $\mathrm{Av}(k\cdots 21)$

$$
\begin{aligned}
(\mathrm{Av}((k-1)\cdots 21), \mathrm{Av}(21))\text{-staircase} &\subseteq \mathrm{Av}(k\cdots 21) \\
&= \mathrm{Av}((k-1)\cdots 21) \odot \mathrm{Av}(21)
\end{aligned}
$$

APPLICATION #1 — $\mathrm{Av}(k\cdots 21)$

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$$\mathrm{gr}(\mathrm{Av}(k\cdots 21)) = \left(\sqrt{(k-2)^2} + \sqrt{1}\right)^2 = (k-1)^2.$$

APPLICATION #2 — $\mathrm{Av}(\alpha \ominus 1 \ominus \gamma)$

Theorem. (Bóna)

$$\mathrm{gr}(\mathrm{Av}(\alpha \ominus 1 \ominus \gamma)) = \left(\sqrt{\mathrm{gr}(\mathrm{Av}(\alpha \ominus 1))} + \sqrt{\mathrm{gr}(\mathrm{Av}(1 \ominus \gamma))}\right)^2.$$

APPLICATION #2 — $\mathrm{Av}(\alpha \ominus 1 \ominus \gamma)$

Theorem. (Bóna)

$$\mathrm{gr}(\mathrm{Av}(\alpha \ominus 1 \ominus \gamma)) = \left(\sqrt{\mathrm{gr}(\mathrm{Av}(\alpha \ominus 1))} + \sqrt{\mathrm{gr}(\mathrm{Av}(1 \ominus \gamma))}\right)^2.$$

New proof.

$$\begin{aligned}(\mathrm{Av}(\alpha \ominus 1), \mathrm{Av}(1 \ominus \gamma))\text{-staircase} &\subseteq \mathrm{Av}(\alpha \ominus 1 \ominus \gamma) \\ &\subseteq \mathrm{Av}(\alpha \ominus 1) \odot \mathrm{Av}(1 \ominus \gamma)\end{aligned}$$

(second subset inequality due to Jelínek and Valtr)

Merge Bound

In every example we know, the upper bound for $\mathrm{gr}(\mathcal{C} \odot \mathcal{D})$ is actually equal to the upper bound.

MERGE BOUND

In every example we know, the upper bound for $\text{gr}(\mathcal{C} \odot \mathcal{D})$ is actually equal to the upper bound.

Are there any classes $\mathcal{C}$ and $\mathcal{D}$ such that

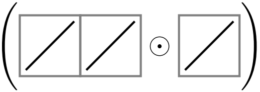
$$\text{gr}(\mathcal{C} \odot \mathcal{D}) < \left(\sqrt{\text{gr}(\mathcal{C})} + \sqrt{\text{gr}(\mathcal{D})}\right)^2?$$

It would require:

- At least one of the classes is neither sum- nor skew-closed and has no sum- or skew-closed subclass with the same growth rate.
- $\mathcal{C}$ and $\mathcal{D}$ have infinite intersection.

Merge Bound

Possible Counterexample?

Does gr  equal $3 + 2\sqrt{2}$?

MERGE BOUND

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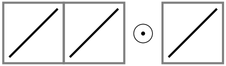
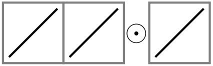
Does gr () equal $3+2\sqrt{2}$?

Probably not useful, but:

 $= \mathrm{Av}(4321, 321654, 421653, 431652, 521643, 531642).$

Main Theorem

Thanks!