

A, B -Minimal Stirling Numbers

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Outline

1 The Basics

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- 4 $\{i\}$ -Minimal Stirling Numbers

Definitions

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$$\min(69/4/38/1257) = \{1, 3, 4, 6\}.$$

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Let $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ denote the *Stirling number of the second kind*, defined by the recursive relationship

$$\left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n \\ k \end{matrix} \right\},$$

with initial conditions $\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = 1$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = 0$ whenever $n < k$ or $n, k < 0$.

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Theorem

For all $n, k \in \mathbb{Z}$, $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = |\Pi_{n,k}|$.

Counting Other Collections of Set Partitions

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- 1 For any set partition π enumerated by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r$, we must have $[r] \subseteq \min(\pi)$.
- 2 This does not exclude $r' > r$ from being a minimal element of a part.

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From looking at the recursion, we find that these count certain m -tuples of set partitions, with the restriction that if π, π' are elements of the same tuple, then $\min(\pi) = \min(\pi')$.

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The two generalizations we illustrated implicitly force some structure on the minimal elements of our parts.

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We call $\mathcal{B}_n(A; B)$ the n -th A, B -minimal Bell number of the second kind.

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We now consider two special cases of minimal A, B-Stirling numbers.

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Theorem

$$\text{For } 1 \leq r < n, S_{n,[r]} = r \sum_{i=0}^{n-r-1} \binom{n-r-1}{i} r^i \mathcal{B}_{n-r-1-i}.$$

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Proof.

Each partition counted by $S_{n,[r]}$ contains $r + 1$ in one of the first r parts, and some $0 \leq i \leq n - r - 1$ of the elements of $\{r + 2, \dots, n\}$ in the first r parts as well, giving the term $r \cdot \binom{n-r-1}{i} \cdot r^i$ for any fixed i .

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Corollary

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$$[[S_{n, [r]}]]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & -4 & 1 & 0 & 0 & 0 \\ -3 & 8 & 9 & 0 & -5 & 1 & 0 & 0 \\ -17 & -8 & 27 & 16 & 0 & -6 & 1 & 0 \\ -33 & -136 & 9 & 64 & 25 & 0 & -7 & 1 \end{bmatrix}$$

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Theorem

For a positive integer n ,
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We say that σ *avoids* the pattern 1-32 if there do not exist $1 \leq i < j < n$ such that $\sigma_i < \sigma_{j+1} < \sigma_j$. We define

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For example, if $\sigma = 968524137$, then $S_1 = 9$, $S_2 = 68$, $S_3 = 5$, $S_4 = 24$, and $S_5 = 137$.

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1234 2134 2314 3124 3214 2413 4123 4213 3412 4312

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$\{i\}$ -Minimal Stirling Numbers

A bijection

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Conjectures

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The End

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Questions?

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