Unknotted Cycles

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Joint with Christopher Cornwell

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Cycle Diagrams

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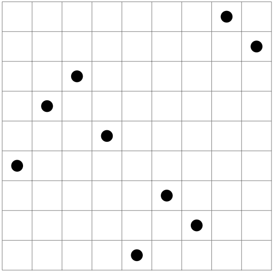

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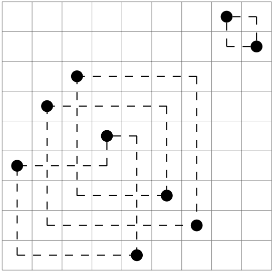

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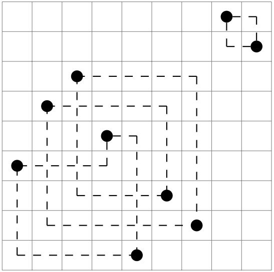

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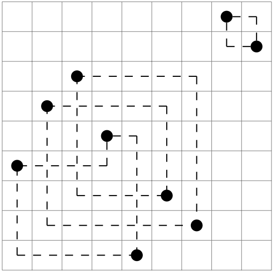

Note: The point $(i, \sigma(i))$ is a corner as is every (i, i) on the diagonal.

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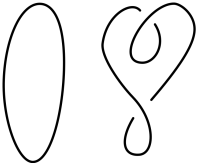

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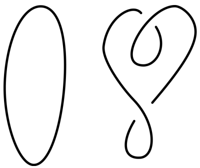

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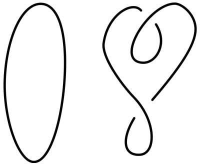

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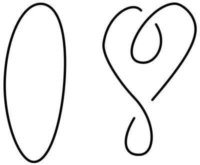

- A **link** is an embedding of one or more circles into $\mathbb{R}^3$.
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- The **unlink** is the link in which every component is the unknot, and none of the components are themselves "linked."

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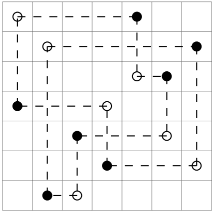

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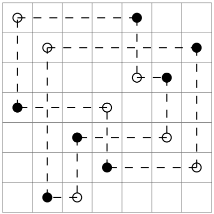

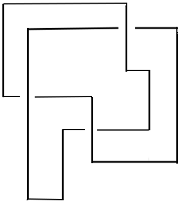

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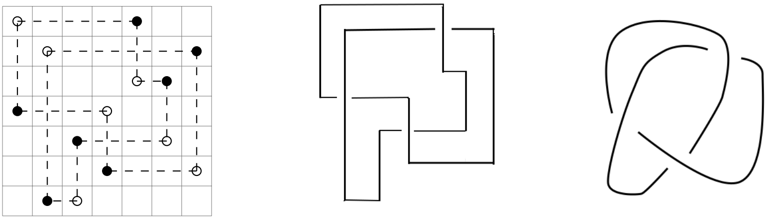

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- There are rules called Cromwell moves that act on grid diagrams without changing the knot type that can transform a grid diagram of a knot into any other grid diagram of the same knot.

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Restricting our attention to *derangements*, permutations without fixed points, every cycle diagram can be interpreted as a grid diagram, associating a link (knot) to each derangement (cycle).

The link associated to a permutation

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For any derangement σ, the **link associated to** σ is obtained by drawing the cycle diagram of σ and interpreting it as a grid diagram instead. If σ is a cycle then this is the **knot associated to** σ.

We will call any cycle associated to the unknot an **unknotted cycle**.

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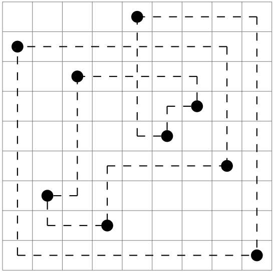

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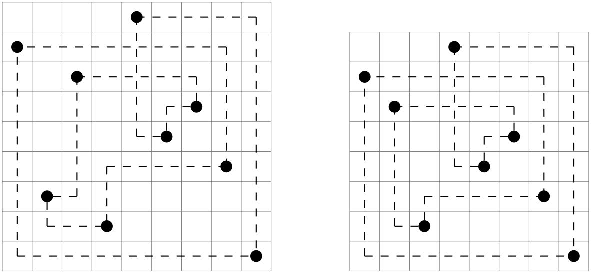

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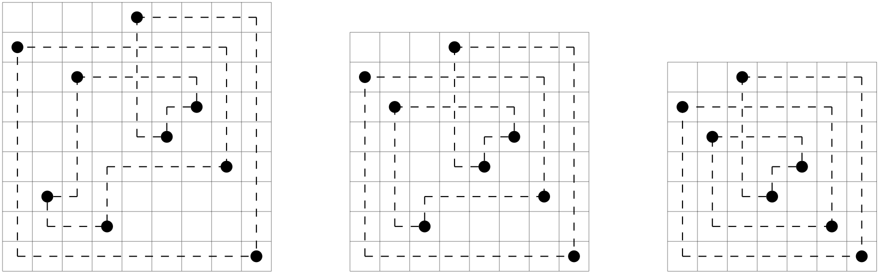

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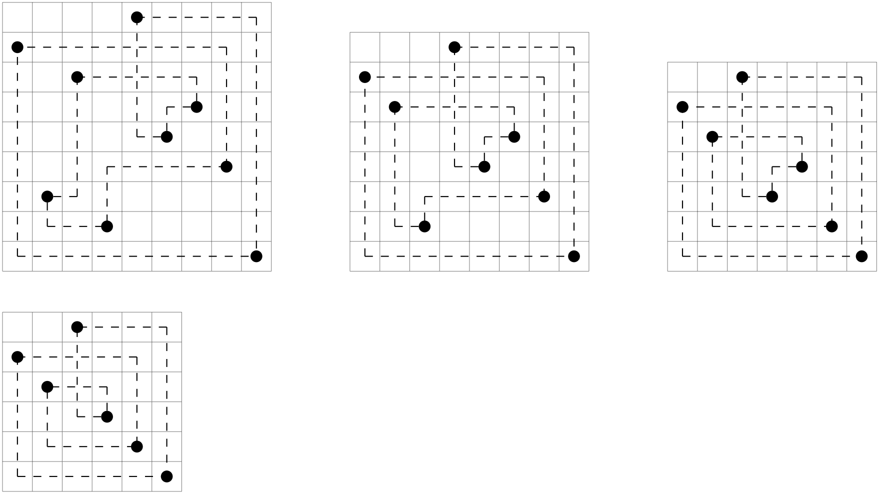

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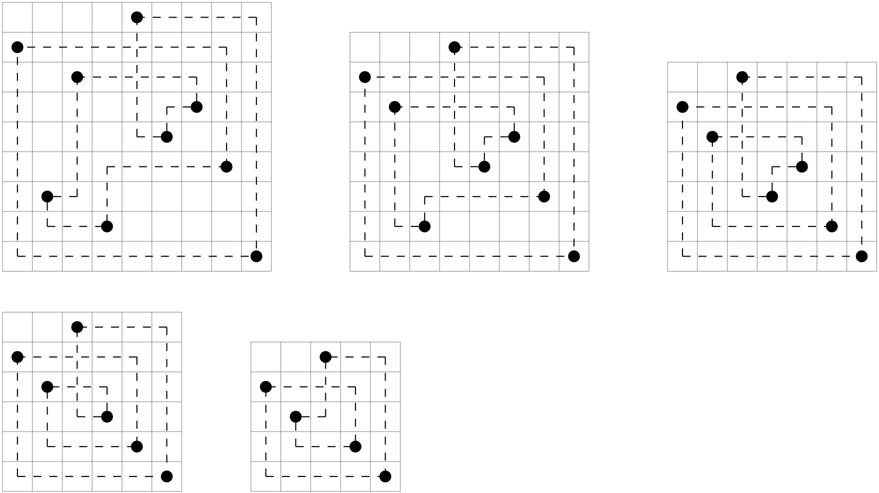

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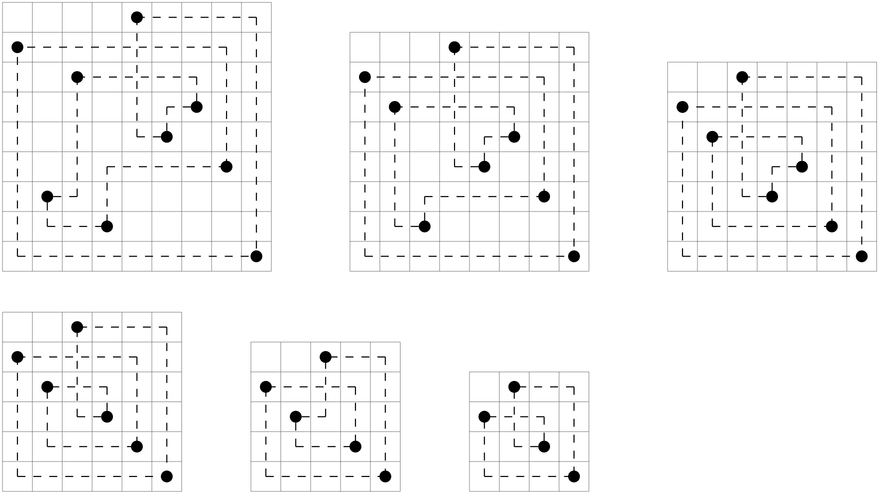

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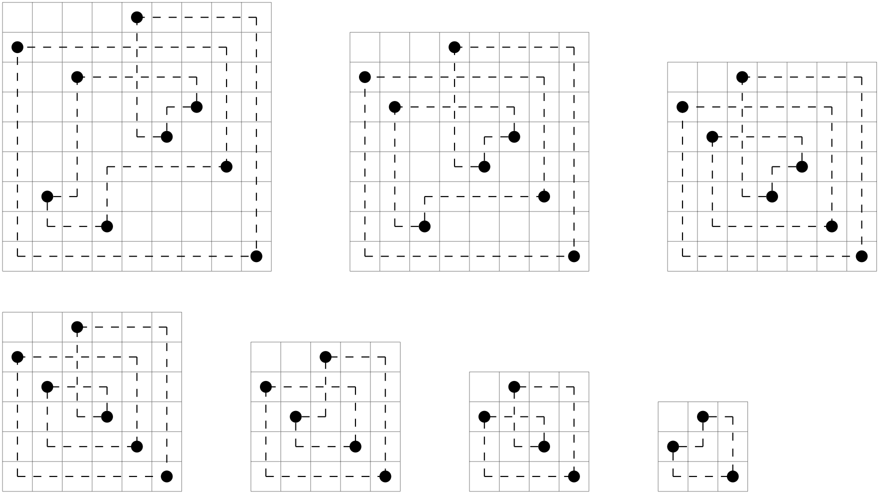

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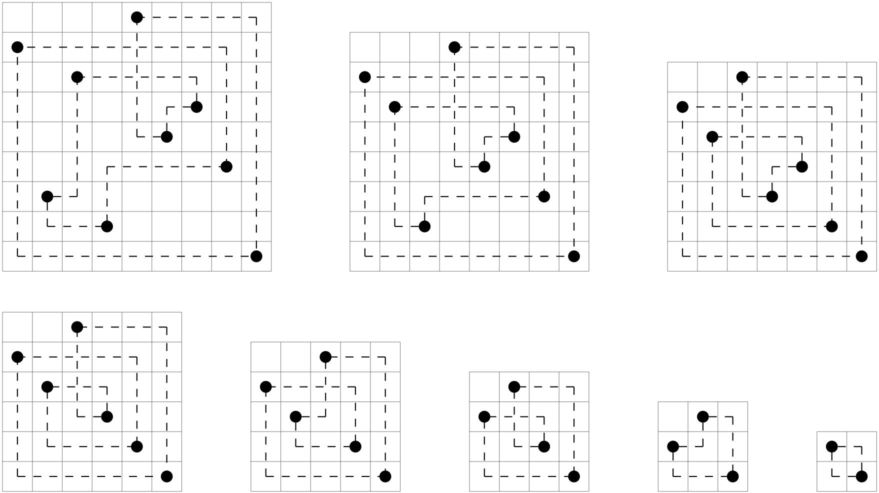

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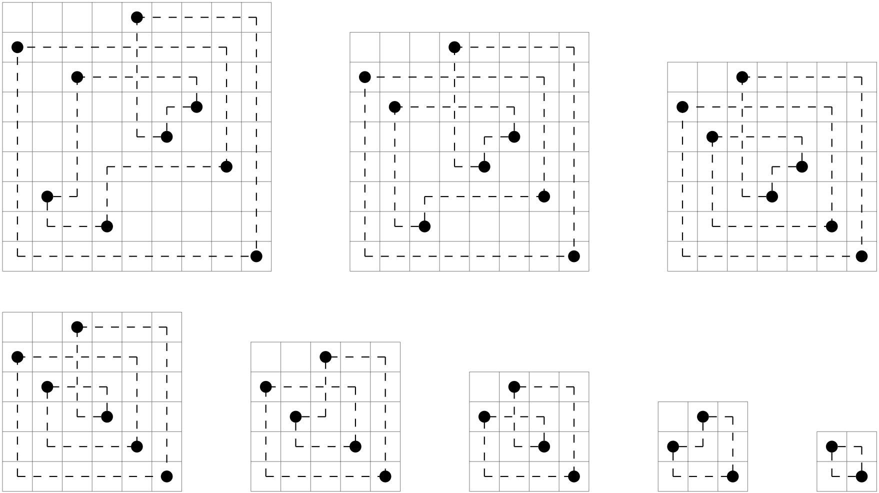

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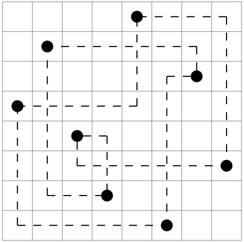

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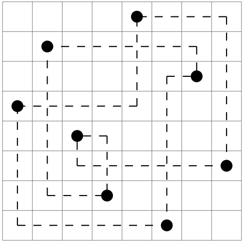

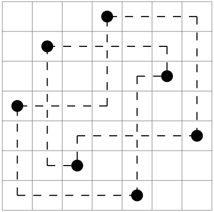

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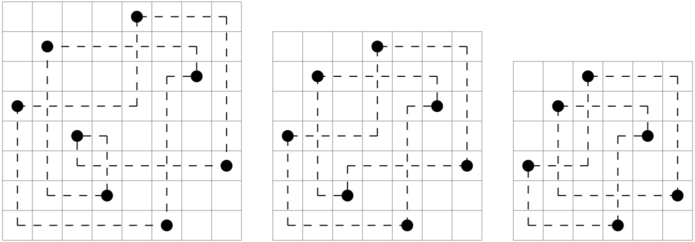

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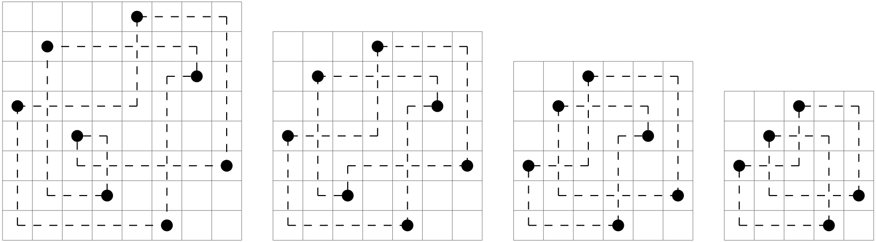

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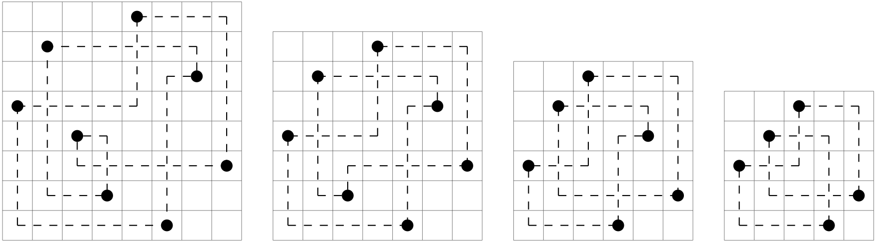

... a trefoil knot.

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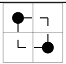
2	1	

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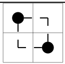
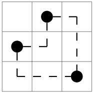
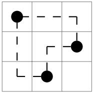
2	1	
3	2	 

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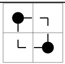
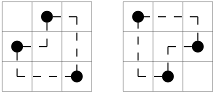
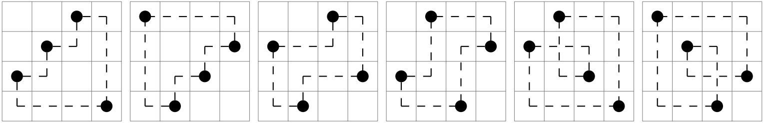
2	1	
3	2	
4	6	

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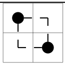
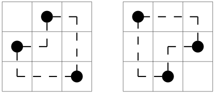
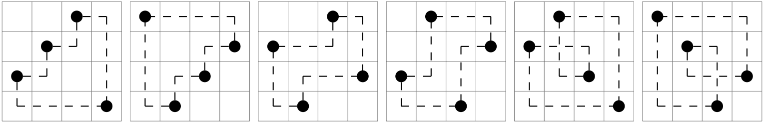
2	1	
3	2	
4	6	
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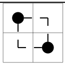
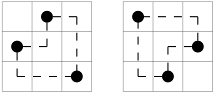
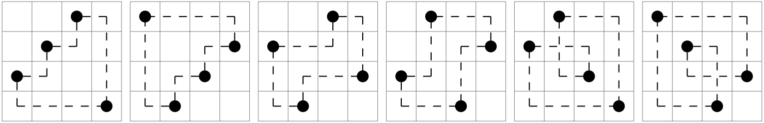
2	1	
3	2	
4	6	
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6	90	…

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Theorem

The count of unknotted cycles of size $n+1$ is S_n.

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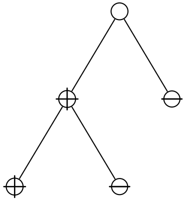
- A node can be rotated into the root. The new node is given the sign of the node rotated into the root.

Example: Rooted-signed-binary-trees



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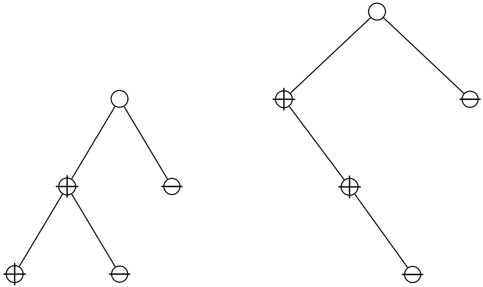

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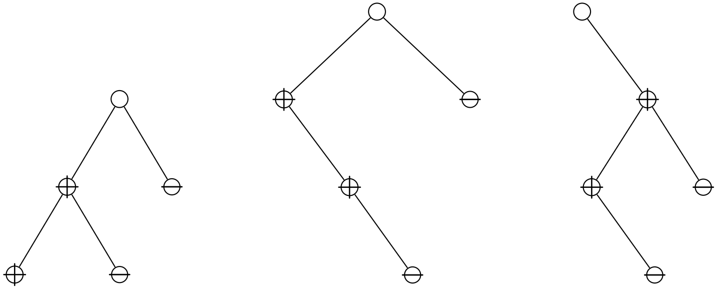

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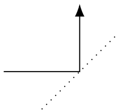
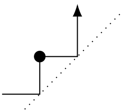
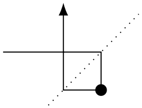
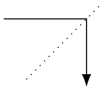
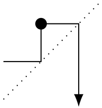
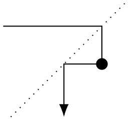
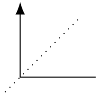
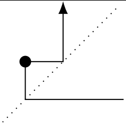
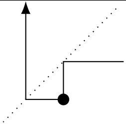
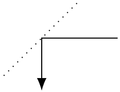
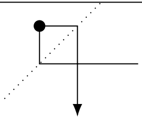
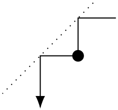
The bijection

Before insertion	After inserting ⊕	After inserting ⊖
		
		
		
		

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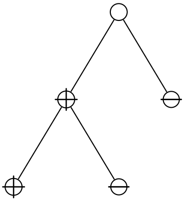

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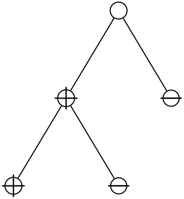

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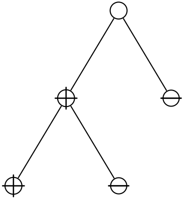

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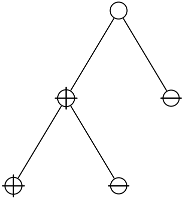

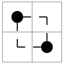

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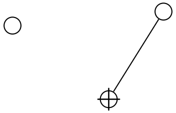

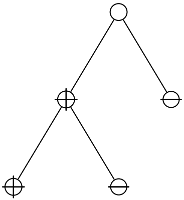

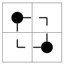

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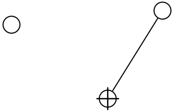

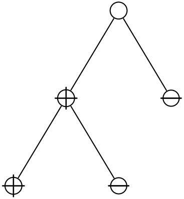

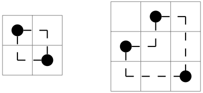

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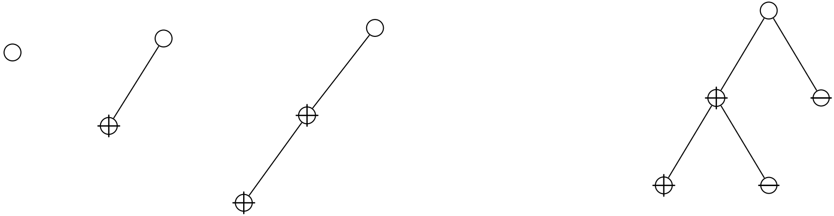

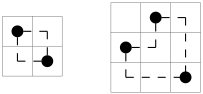

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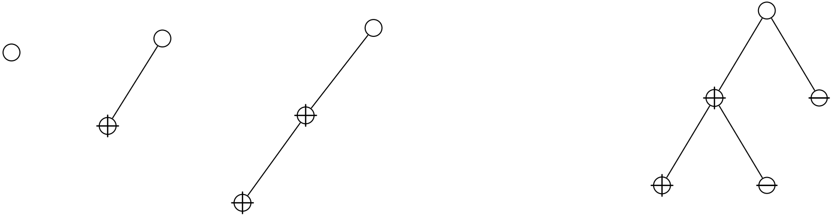

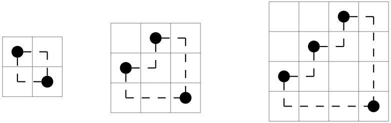

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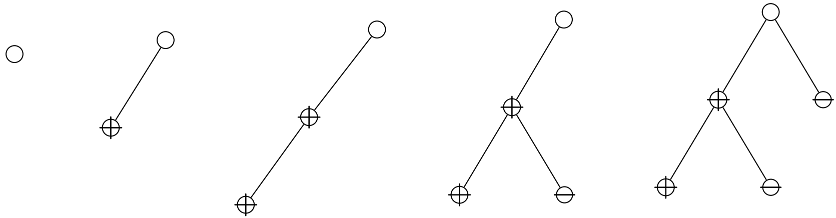

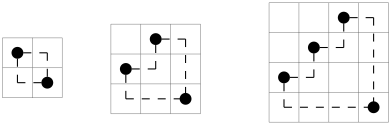

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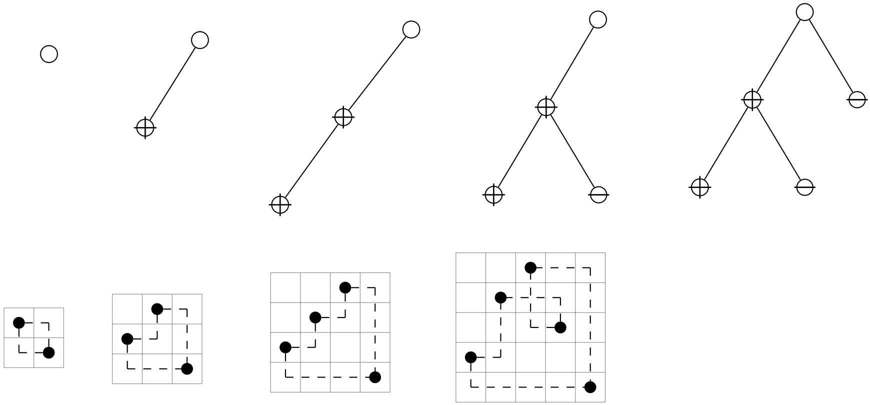

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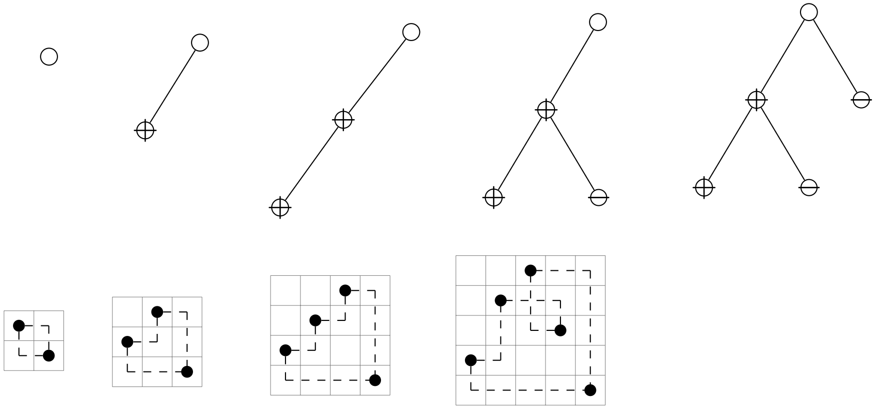

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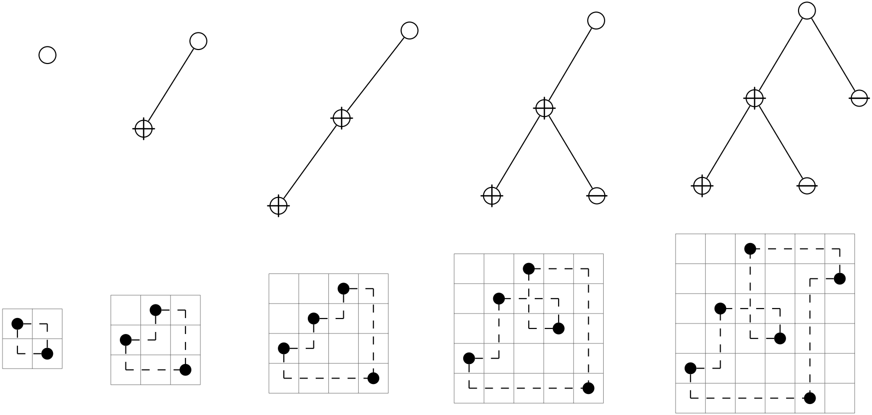

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To show surjectivity, it would suffice to show that every unknotted cycle, σ, has a point on the off-diagonal, i.e $|\sigma(i) - i| = 1$.

Topology

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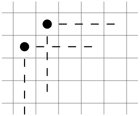

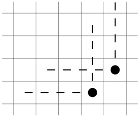

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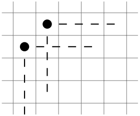

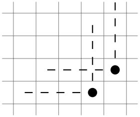

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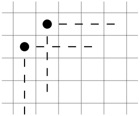

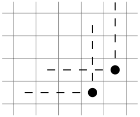

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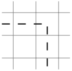

Topology

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Let σ be a cycle of length at least 2, K the knot associated to σ, and $g(K)$ the Seifert genus of K. Then

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If σ is a cycle with $|\sigma(i) - i| \geq 2$ for all i, then each upper right corner of σ corresponds to a unique crossing of σ. So $C(\sigma) \geq UR(\sigma)$.

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If σ is an unknotted cycle, then $|\sigma(i) - i| = 1$ for at least one index i.

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Topology

Corollary

The map from rooted-signed-binary trees to unknotted cycles is surjective.

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If σ is an unlinked derangement then no crossing in the cycle diagram of σ is between different components of the link.

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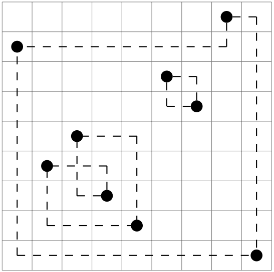
$$\sigma = 845327691$$



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or equivalently

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Counting Unlinks

Observations:

- The coefficient of u in $F(u,x)$ is $\frac{1}{2}(x - x^2 - x\sqrt{x^2 - 6x + 1})$, the (shifted) generating function of the large Schröder numbers.

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- Count *any* knot/link type besides the unknot/unlink.
  - Trefoils: 0,0,0,2,28,264,2098, 15176 . . .
  - It would suffice to count primitive trefoil cycles (those without points on the off diagonal): 0,0,0,2,10,34,96,248 . . .
- What is the most common knot type for a given cycle length? Unlink dominates for $2 \leq n \leq 8$, trefoil dominates for $n = 9, 10, 11 \ldots$
- Can we classify all knots that are associated to some cycle?
  - Must be negative knots.
  - So far only negative braid knots and connected sums of negative braid knots have been observed.

Thank you!