

Prolific Permutations and Expected Breadth

Cheyne Homberger

Permutation Patterns 2018

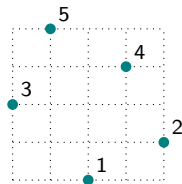
Joint work with Simon Blackburn and Pete Winkler

Plotting Permutations

Definition

If π is a permutation of length n , then the **plot** of π is the set of points

$$\{(1, \pi(1)), (2, \pi(2)), \dots, (n, \pi(n))\} \subset \mathbb{R}^2$$

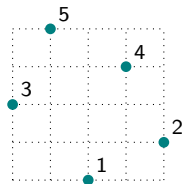


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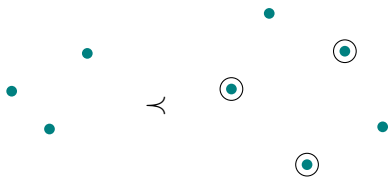


$$\pi = 35142$$

Permutation Patterns

Definition

Let $\pi = \pi(1)\pi(2)\cdots\pi(n)$ and $\sigma = \sigma(1)\sigma(2)\cdots\sigma(k)$ be two permutations. π **contains** σ **as a pattern** (written $\sigma \prec \pi$) if there is some subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ which is order isomorphic to the entries of σ (i.e., $\pi(i_j) < \pi(i_k)$ if and only if $\sigma(j) < \sigma(k)$).



213

\prec

35142

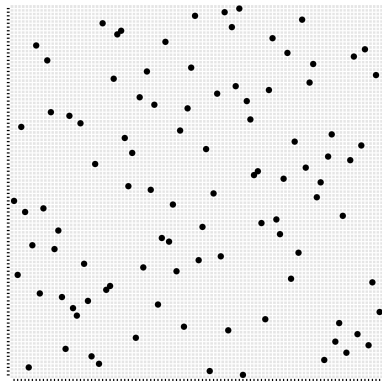
Permutation Breadth

The **breadth** of a permutation π is

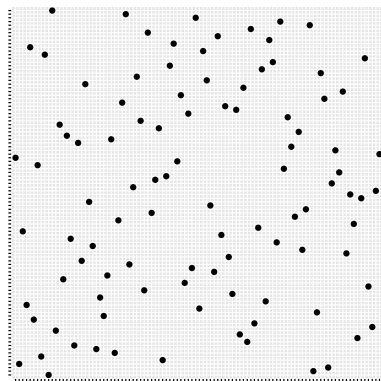
$$\min_{i \neq j} \left\{ |\pi(i) - \pi(j)| + |i - j| \right\}.$$

Permutation Breadth

The **breadth** of a permutation is the minimum pairwise manhattan distance between entries of its plot.

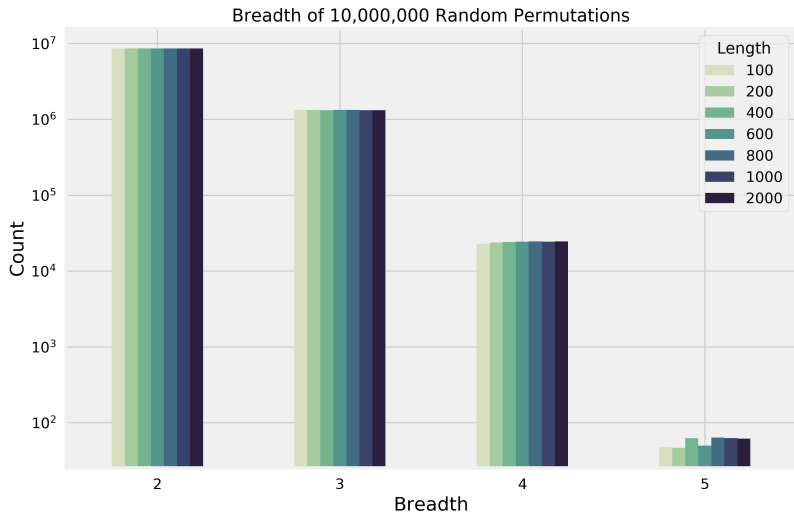


Breadth 2

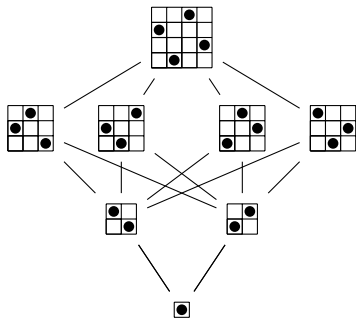
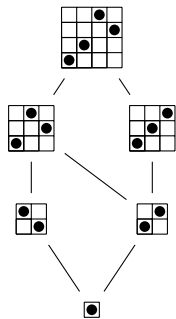
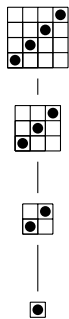


Breadth 4

Random Permutations



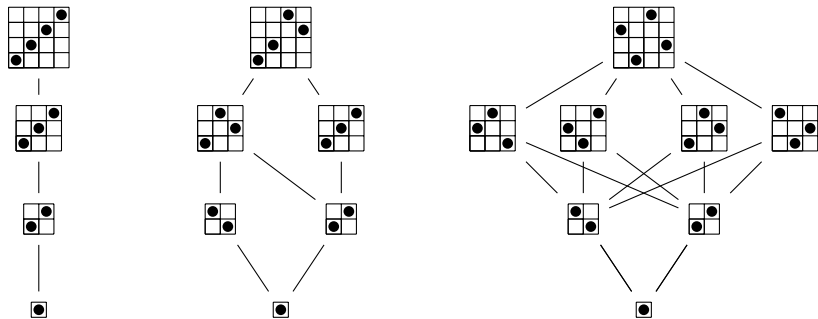
Principal Downsets and Prolific Permutations



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A permutation π of length n is **d -prolific** if each $n - d$ subset of entries forms a unique pattern. That is, the principal downset generated by π is as $\binom{n}{d}$ -wide at rank $n - d$.



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1-prolific

Characterizing Prolificity

Theorem (Bevan, H., Tenner)

A permutation is d -prolific if and only if its has breadth $\geq d + 2$.

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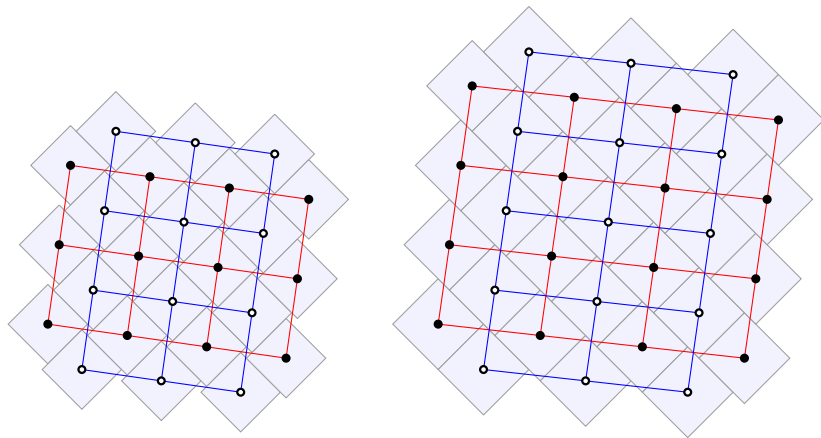
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Theorem (Bevan, H., Tenner)

There exists an n -permutation with breadth $\geq d + 2$ if and only if

$$n \geq d^2/2 + 2d + 1.$$

Maximum Breadth, Minimum Length



5-prolific 24-permutation and 6-prolific 31-permutation.

The Prolific Portion of Permutations

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As $n \rightarrow \infty$, a random n -permutation has breadth $\geq d + 2$, (and hence is d -prolific) with probability approaching

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Theorem (Blackburn, H., Winkler)

The expected breadth of a random permutation approaches

$$\mathbb{E}[\mathbf{br}] \rightarrow 1 + \sum_{d \geq 0} e^{-d^2 - d} \approx 2.13782018 \dots$$

Idea: Close Pairs

Fix d, n .

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Note

A permutation has breadth $\geq d + 2$ (and hence is d -prolific) iff it has no close pairs.

Idea of Proof

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For $1 \leq i \leq n$, let X_i be the indicator random variable for the pair i, j being a close pair for some $j > i$.

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Key Idea

Most X_i are **mostly** independent, $\sum_i X_i$ is **close** to the number of close pairs, and for **most** i , we have

$$\mathbb{P}[X_i] \approx \lambda/n$$

with $\lambda := 2\binom{d+1}{2} = d^2 + d$.

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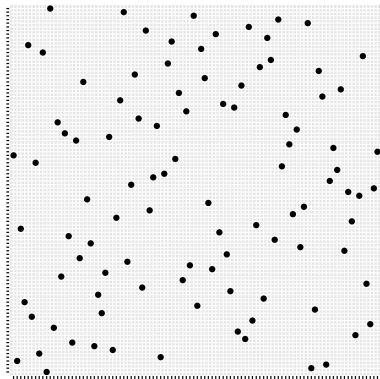
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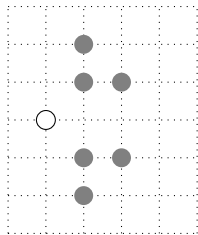
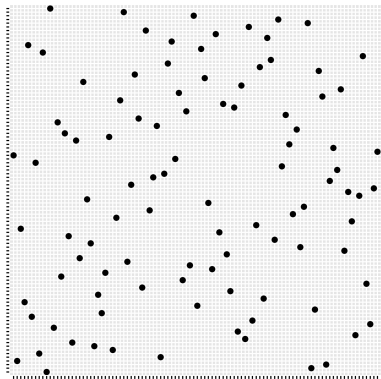
Fact (Poisson Approximation)

If we have n independent and identically distributed Bernoulli random variables each with mean λ/n , then, as $n \rightarrow \infty$, their sum is Poisson- λ .

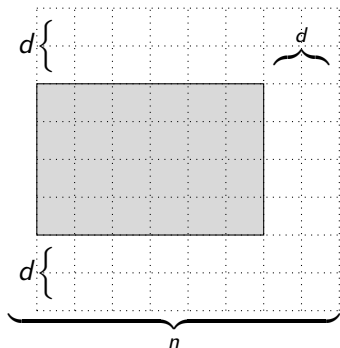
Example



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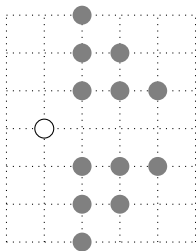


$$\mathbb{P}[X_i = 1]$$



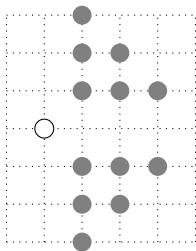
Shaded region has $\sim n^2$ entries, while remainder has $\sim n$.

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In general we have $\lambda := 2\binom{d+1}{2} = d^2 + d$ potential spots which can make i into the start of a close pair.

Theorem

$$\mathbb{P}[X_i = 1] = \frac{\lambda}{n} + O(n^{-2}).$$

Rough Sketch

Strategy (Wolfowitz, 1944)

Let $X = \sum_{i=1}^n X_i$. Let Y_i be iid Bernoulli random variables with mean λ , and let $Y := \sum_{i=1}^n Y_i$.

We will show that

$$\lim_{n \rightarrow \infty} \mathbb{E} [X^t] = \lim_{n \rightarrow \infty} \mathbb{E} [Y^t].$$

Then, since $Y := \sum_{i=1}^n Y_i$ is asymptotically Poisson, we have that X is Poisson with mean λ , which completes our proof.

Distribution of Close Pairs

Theorem

For fixed d , the distribution of d -close pairs in a random n -permutation is Poisson with mean λ .

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Corollary

The expected number of permutations with no d -close pairs is

$$e^{-\lambda} = e^{-d^2-d}.$$

Expected Breadth

Caveat

Knowing the asymptotic distribution of close pairs for a given d is **not strong enough** to calculate the expected breadth of a random permutation.

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Expectation

Let \mathbf{br} be the breadth of the random permutation π . Then

$$\begin{aligned}\mathbb{E}[\mathbf{br}] &= 2 \cdot \mathbb{P}[\mathbf{br} = 2] + 3 \cdot \mathbb{P}[\mathbf{br} = 3] + 4 \cdot \mathbb{P}[\mathbf{br} = 4] + \dots \\ &= 1 + \left(\underbrace{\mathbb{P}[\mathbf{br} \geq 2]}_{\rightarrow e^0} + \underbrace{\mathbb{P}[\mathbf{br} \geq 3]}_{\rightarrow e^{-2}} + \underbrace{\mathbb{P}[\mathbf{br} \geq 4]}_{\rightarrow e^{-6}} + \dots \right)\end{aligned}$$

Boole's Inequality and the Bonferroni Inequalities

Let $\{A_i\}_{i=1}^n$ be events. Then

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Also, letting

$$S_1 := \sum_i \mathbb{P}[A_i], \quad S_2 := \sum_{i < j} \mathbb{P}[A_i \cap A_j],$$

$$S_k := \sum_{i_1 < i_2 < \dots < i_k} \mathbb{P}[A_{i_1} \cap \dots \cap A_{i_k}],$$

we have, for all k ,

$$\sum_{j=1}^{2k} (-1)^{j-1} S_j \leq \mathbb{P}\left[\bigcup_{i=1}^n A_i\right] \leq \sum_{j=1}^{2k+1} (-1)^{j-1} S_j.$$

Applying the Bonferroni Inequalities

Let X_i be the indicator variable for $\pi(i)$ being the initial entry in a close pair. Then

$$\mathbb{P} \left[\bigcup_i \{X_i = 1\} \right]$$

is the probability that π has breadth $\geq d$.

Therefore

$$\sum_{j=1}^{2k} (-1)^{j-1} S_j \leq \mathbb{P} [\mathbf{br} \geq d] \leq \sum_{j=1}^{2k+1} (-1)^{j-1} S_j,$$

where

$$S_k = \sum_{i_1 < \dots < i_k} \mathbb{E} [X_{i_1} \cdots X_{i_k}].$$

Direct Proof

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Lemma

Let d be a function of n such that $d = O(\log n)$. The probability that a uniformly chosen permutation π has breadth d is

$$e^{-\lambda} + O((\log n)^6 e^\lambda / n).$$

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The **minimum jump (mj)** of a permutation is the biggest distance between two adjacent entries of a permutation. It turns out this is easier to calculate, and translates well to breadth.

Lemma

Fix a positive integer t . There exist functions $\{p_i(d)\}_{i=1}^t$ which are all at most polynomial, such that if $d = O(\log n)$, then

$$\mathbb{P}[\mathbf{mj} > d] = \left(1 + \sum_{i=1}^{t-1} \frac{p_i(d)}{n^i}\right) e^{-2d} + O\left(\frac{p_t(d)}{n^t} e^{2d}\right)$$

(Very Rough) Sketch

Let $\mathbf{br}(\pi)$ be the breadth of the random permutation π .

$$\begin{aligned}\mathbb{E}[\mathbf{br}(\pi)] &= -1 + \sum_{d=0}^{\lceil\sqrt{2n}\rceil} \mathbb{P}[\mathbf{br}(\pi) \geq d] \\ &= -1 + \sum_{d=0}^{\lceil(\log n)/2\rceil} \mathbb{P}[\mathbf{br}(\pi) \geq d] \\ &\quad + \sum_{d=\lceil(\log n)/2\rceil}^{\lceil\log n\rceil} \mathbb{P}[\mathbf{br}(\pi) \geq d] \\ &\quad + \sum_{d=\lceil\log n\rceil}^{\lceil\sqrt{2n}\rceil} \mathbb{P}[\mathbf{br}(\pi) \geq d]. \\ &= 2.13782018\dots\end{aligned}$$

Conclusion

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Theorem

The distribution of close pairs is Poisson- λ with $\lambda = d^2 + d$, and so a random n -permutation has breadth $\geq d$ with probability approaching

$$e^{-d^2-d}.$$

Theorem

The expected breadth of a random permutation approaches

$$1 + \sum_{d \geq 0} e^{-d^2-d} \approx 2.13782018 \dots$$

The expected minimum jump of a permutation is

$$\sum_{d \geq 0} e^{-2d} \approx 1.156517 \dots$$

Complexity

Further, these ideas lead to an algorithm for measuring breadth which is $O(n^{3/2})$ in the worst case, and $O(n)$ on average.

Thanks!