

Shuffle-compatibility for the exterior peak set

Darij Grinberg (UMN)

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Dartmouth College

slides: <http://www.cip.ifi.lmu.de/~grinberg/algebra/dartmouth18.pdf>

paper: <http://www.cip.ifi.lmu.de/~grinberg/algebra/gzshuf2.pdf>

project: <https://github.com/darijgr/gzshuf>

Section 1

Shuffle-compatibility

Reference:

- Ira M. Gessel, Yan Zhuang, *Shuffle-compatible permutation statistics*, arXiv:1706.00750, Adv. in Math. **332** (2018), pp. 85–141.

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- For $n \in \mathbb{N}$, an *n -permutation* means an n -tuple of distinct positive integers (“letters”).
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- A *permutation* means an n -permutation for some n .
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We say that π is *nonempty* if $n > 0$.
- If π is an n -permutation and $i \in \{1, 2, \dots, n\}$, then π_i denotes the i -th entry of π .

- Two n -permutations α and β (with the same n) are *order-equivalent* if all $i, j \in \{1, 2, \dots, n\}$ satisfy $(\alpha_i < \alpha_j) \iff (\beta_i < \beta_j)$.
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Note. A statistic need not be integer-valued! It can be set-valued, or list-valued for example.

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Examples of permutation statistics, 1: descents et al

- If π is an n -permutation, then a *descent* of π means an $i \in \{1, 2, \dots, n-1\}$ such that $\pi_i > \pi_{i+1}$.
- The *descent set* $\text{Des } \pi$ of a permutation π is the set of all descents of π .

Thus, Des is a statistic.

Example: $\text{Des}(3, 1, 5, 2, 4) = \{1, 3\}$.

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- The *descent number* $\text{des } \pi$ of a permutation π is the number of all descents of π : that is, $\text{des } \pi = |\text{Des } \pi|$.

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- The *major index* $\text{maj } \pi$ of a permutation π is the **sum** of all descents of π .

Thus, maj is a statistic.

Example: $\text{maj}(3, 1, 5, 2, 4) = 1 + 3 = 4$.

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Thus, maj is a statistic.

Example: $\text{maj}(3, 1, 5, 2, 4) = 1 + 3 = 4$.

- The *Coxeter length* inv (i.e., *number of inversions*) and the *set of inversions* are statistics, too.

Examples of permutation statistics, 2: peaks

- If π is an n -permutation, then a *peak* of π means an $i \in \{2, 3, \dots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$.
(Thus, peaks can only exist if $n \geq 3$.)

The name refers to the plot of π , where peaks look like this:

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(Thus, peaks can only exist if $n \geq 3$.)

The name refers to the plot of π , where peaks look like this:
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- The *peak set* $Pk\pi$ of a permutation π is the set of all peaks of π .

Thus, Pk is a statistic.

Examples:

- $Pk(3, 1, 5, 2, 4) = \{3\}$.
- $Pk(1, 3, 2, 5, 4, 6) = \{2, 4\}$.
- $Pk(3, 2) = \{\}$.

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 - $Pk(1, 3, 2, 5, 4, 6) = \{2, 4\}$.
 - $Pk(3, 2) = \{\}$.
 - The *peak number* $pk \pi$ of a permutation π is the number of all peaks of π : that is, $pk \pi = |Pk \pi|$.
Thus, pk is a statistic.
- Example:** $pk(3, 1, 5, 2, 4) = 1$.

Examples of permutation statistics, 3: left peaks

- If π is an n -permutation, then a *left peak* of π means an $i \in \{1, 2, \dots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$.
(Thus, left peaks are the same as peaks, except that 1 counts as a left peak if $\pi_1 > \pi_2$.)
- The *left peak set* $\text{Lpk } \pi$ of a permutation π is the set of all left peaks of π .

Thus, Lpk is a statistic.

Examples:

- $\text{Lpk}(3, 1, 5, 2, 4) = \{1, 3\}$.
- $\text{Lpk}(1, 3, 2, 5, 4, 6) = \{2, 4\}$.
- $\text{Lpk}(3, 2) = \{1\}$.
- The *left peak number* $\text{lpk } \pi$ of a permutation π is the number of all left peaks of π : that is, $\text{lpk } \pi = |\text{Lpk } \pi|$.

Thus, lpk is a statistic.

Example: $\text{lpk}(3, 1, 5, 2, 4) = 2$.

Examples of permutation statistics, 4: right peaks

- If π is an n -permutation, then a *right peak* of π means an $i \in \{2, 3, \dots, n\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_{n+1} = 0$.
(Thus, right peaks are the same as peaks, except that n counts as a right peak if $\pi_{n-1} < \pi_n$.)
- The *right peak set* $\text{Rpk } \pi$ of a permutation π is the set of all right peaks of π .
Thus, Rpk is a statistic.

Examples:

- $\text{Rpk}(3, 1, 5, 2, 4) = \{3, 5\}$.
 - $\text{Rpk}(1, 3, 2, 5, 4, 6) = \{2, 4, 6\}$.
 - $\text{Rpk}(3, 2) = \{\}$.
 - The *right peak number* $\text{rpk } \pi$ of a permutation π is the number of all right peaks of π : that is, $\text{rpk } \pi = |\text{Rpk } \pi|$.
Thus, rpk is a statistic.
- Example:** $\text{rpk}(3, 1, 5, 2, 4) = 2$.

Examples of permutation statistics, 5: exterior peaks

- If π is an n -permutation, then an *exterior peak* of π means an $i \in \{1, 2, \dots, n\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$ and $\pi_{n+1} = 0$.
(Thus, exterior peaks are the same as peaks, except that 1 counts if $\pi_1 > \pi_2$, and n counts if $\pi_{n-1} < \pi_n$.)

- The *exterior peak set* $\text{Epk } \pi$ of a permutation π is the set of all exterior peaks of π .

Thus, Epk is a statistic.

Examples:

- $\text{Epk}(3, 1, 5, 2, 4) = \{1, 3, 5\}$.
- $\text{Epk}(1, 3, 2, 5, 4, 6) = \{2, 4, 6\}$.
- $\text{Epk}(3, 2) = \{1\}$.
- Thus, $\text{Epk } \pi = \text{Lpk } \pi \cup \text{Rpk } \pi$ if $n \geq 2$.
- The *exterior peak number* $\text{epk } \pi$ of a permutation π is the number of all exterior peaks of π : that is, $\text{epk } \pi = |\text{Epk } \pi|$.
Thus, epk is a statistic.

Example: $\text{epk}(3, 1, 5, 2, 4) = 3$.

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- We let $S(\pi, \sigma)$ be the set of all shuffles of π and σ .
- **Example:**

$$S((4, 1), (2, 5)) = \{(4, 1, 2, 5), (4, 2, 1, 5), (4, 2, 5, 1), \\ (2, 4, 1, 5), (2, 4, 5, 1), (2, 5, 4, 1)\}.$$

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- Observe that π and σ have $\binom{m+n}{m}$ shuffles, in bijection with m -element subsets of $\{1, 2, \dots, m + n\}$.

Shuffle-compatible statistics: definition

- A statistic st is said to be *shuffle-compatible* if for any two disjoint permutations π and σ , the multiset

$$\{st \tau \mid \tau \in S(\pi, \sigma)\}_{\text{multiset}}$$

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In particular, it has to stay unchanged if π and σ are replaced by two permutations order-equivalent to them: e.g., st must have the same distribution on the three sets

$$S((4, 1), (2, 5)), \quad S((2, 1), (3, 5)), \quad S((9, 8), (2, 3)).$$

Shuffle-compatible statistics: results of Gessel and Zhuang

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- Their proofs use a mixture of enumerative combinatorics (including some known formulas of MacMahon, Stanley, ...), quasisymmetric functions, Hopf algebra theory, P-partitions (and variants by Stembridge and Petersen), Eulerian polynomials (based on earlier work by Zhuang, and even earlier work by Foata and Strehl).

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- **Theorem (G.)**. The statistic Epk is shuffle-compatible (as conjectured in Gessel/Zhuang).

LR-shuffle-compatibility

- We further introduce a finer version of shuffle-compatibility: “LR-shuffle-compatibility” .
- Given two disjoint nonempty permutations π and σ ,
 - a *left shuffle* of π and σ is a shuffle of π and σ that starts with a letter of π ;
 - a *right shuffle* of π and σ is a shuffle of π and σ that starts with a letter of σ .

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- We let $S_{\prec}(\pi, \sigma)$ be the set of all *left* shuffles of π and σ . We let $S_{\succ}(\pi, \sigma)$ be the set of all *right* shuffles of π and σ .
- A statistic st is said to be *LR-shuffle-compatible* if for any two disjoint nonempty permutations π and σ , the multisets $\{st \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multiset}}$ and $\{st \tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multiset}}$ depend only on $st \pi$, $st \sigma$, $|\pi|$, $|\sigma|$ and the truth value of $\pi_1 > \sigma_1$.

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- **Theorem (G.)**. Des, des, Lpk and Epk are LR-shuffle-compatible.

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- **Theorem (G.)**. Des, des, Lpk and Epk are LR-shuffle-compatible. (But not maj or Rpk or Pk.)

LR-shuffle-compatibility: alternative definition

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- The “LR” in “LR-shuffle-compatibility” stands for “left and right”. Indeed:
- A statistic st is said to be *left-shuffle-compatible* if for any two disjoint nonempty permutations π and σ such that

$$\pi_1 > \sigma_1,$$

the multiset

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depends only on $st \pi$, $st \sigma$, $|\pi|$ and $|\sigma|$.

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- **Proposition.** A permutation statistic st is LR-shuffle-compatible if and only if it is both left-shuffle-compatible and right-shuffle-compatible.

Section 2

Methods of proof

References:

- Darij Grinberg, *Shuffle-compatible permutation statistics II: the exterior peak set*.
- John R. Stembridge, *Enriched P -partitions*, Trans. Amer. Math. Soc. 349 (1997), no. 2, pp. 763–788.
- T. Kyle Petersen, *Enriched P -partitions and peak algebras*, Adv. in Math. 209 (2007), pp. 561–610.

- Now to the general ideas of our proof that E_{pk} is shuffle-compatible.
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- More precisely, we define **\mathcal{Z} -enriched P -partitions**: a generalization of
 - P -partitions (Stanley 1972);
 - enriched P -partitions (Stembridge 1997);
 - left enriched P -partitions (Petersen 2007),which are used in the proofs for Des , P_k and L_{pk} , respectively.

- Now to the general ideas of our proof that E_{pk} is shuffle-compatible.
- Strategy: imitate the classical proofs for Des , P_k and L_{pk} , using (yet) another version of enriched P -partitions.
- More precisely, we define **\mathcal{Z} -enriched P -partitions**: a generalization of
 - P -partitions (Stanley 1972);
 - enriched P -partitions (Stembridge 1997);
 - left enriched P -partitions (Petersen 2007),which are used in the proofs for Des , P_k and L_{pk} , respectively.
- The idea is simple, but the proof takes work. Let me just show the highlights without using P -partition language.

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- If $n \in \mathbb{N}$ and if Λ is any subset of $[n]$, then we define a power series $K_{n,\Lambda}^{\mathcal{Z}} \in \text{Pow } \mathcal{N}$ by

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- **Product formula.** If π is an n -permutation and σ is an m -permutation, then

$$K_{n, \text{Epk } \pi}^{\mathbb{Z}} \cdot K_{m, \text{Epk } \sigma}^{\mathbb{Z}} = \sum_{\tau \in \mathcal{S}(\pi, \sigma)} K_{n+m, \text{Epk } \tau}^{\mathbb{Z}}$$

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- Proof idea: $K_{n,\text{Epk } \pi}^{\mathcal{Z}}$ is the generating function of \mathcal{Z} -enriched P -partitions for a certain totally ordered set P .

Lacunar subsets and linear independence

- A set S of integers is called *lacunar* if it contains no two consecutive integers. (Some call this “sparse”.)
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- **Lemma.** For each nonempty permutation π , the set $\text{Epk } \pi$ is a nonempty lacunar subset of $[n]$.
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(And conversely – although we don’t need it –, any such subset has the form $\text{Epk } \pi$ for some π .)
- **Lemma.** The family

$$\left(K_{n,\Lambda}^{\mathbb{Z}} \right)_{n \in \mathbb{N}; \Lambda \subseteq [n] \text{ is lacunar and nonempty}}$$

is \mathbb{Q} -linearly independent.

- These lemmas, and the above product formula, prove the shuffle-compatibility of Epk .

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- A statistic st is said to be **LR-shuffle-compatible** if for any two disjoint nonempty permutations π and σ , the multisets

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- We claim that Des, des, Lpk and Epk are LR-shuffle-compatible.

Head-graft-compatibility

- Crucial observation:

(LR-shuffle-compatible)

\iff (shuffle-compatible) \wedge (head-graft-compatible).

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- A permutation statistic st is said to be *head-graft-compatible* if for any nonempty permutation π and any letter a that does not appear in π , the element $st(a : \pi)$ depends only on $st(\pi)$, $|\pi|$ and on the truth value of $a > \pi_1$.
Here, $a : \pi$ is the permutation obtained from π by appending a at the front:

$$\pi = (\pi_1, \pi_2, \dots, \pi_n) \quad \implies \quad a : \pi = (a, \pi_1, \pi_2, \dots, \pi_n).$$

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- For example, Epk is head-graft-compatible, since

$$\text{Epk}(a : \pi) = \begin{cases} \text{Epk } \pi + 1, & \text{if not } a > \pi_1; \\ ((\text{Epk } \pi + 1) \setminus \{2\}) \cup \{1\}, & \text{if } a > \pi_1. \end{cases}$$

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- Likewise, Des , Lpk and des are head-graft-compatible.

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- **Theorem (G.).** A statistic st is LR-shuffle-compatible **if and only if** it is shuffle-compatible and head-graft-compatible.

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- **Theorem (G.).** A statistic st is LR-shuffle-compatible **if and only if** it is shuffle-compatible and head-graft-compatible.
- Hence, Epk , Des , Lpk and des are LR-shuffle-compatible.

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If π is an n -permutation with $n > 0$, then let $\pi_{\sim 1}$ be the $(n - 1)$ -permutation $(\pi_2, \pi_3, \dots, \pi_n)$.

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If π and σ are two disjoint permutations, then

$$S_{\prec}(\pi, \sigma) = S_{\succ}(\sigma, \pi);$$

$$S_{\prec}(\pi, \sigma) = S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma) \quad \text{if } \pi \text{ is nonempty};$$

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These allow for an inductive argument.

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These allow for an inductive argument.

- Note that the concept of LR-shuffle-compatibility is not invariant under reversal: st can be LR-shuffle-compatible while $st \circ \text{rev}$ is not, where

$$\text{rev}(\pi_1, \pi_2, \dots, \pi_n) = (\pi_n, \pi_{n-1}, \dots, \pi_1).$$

For example, Lpk is LR-shuffle-compatible, but Rpk is not.

Section 3

The QSym connection

References:

- Ira M. Gessel, Yan Zhuang, *Shuffle-compatible permutation statistics*, arXiv:1706.00750.
- Darij Grinberg, Victor Reiner, *Hopf Algebras in Combinatorics*, arXiv:1409.8356, and various other texts on combinatorial Hopf algebras.

- Gessel and Zhuang prove **most** of their shuffle-compatibilities algebraically. Their methods involve combinatorial Hopf algebras (QSym and NSym).
- These methods work for **descent statistics** only. What is a descent statistic?

- Gessel and Zhuang prove **most** of their shuffle-compatibilities algebraically. Their methods involve combinatorial Hopf algebras (QSym and NSym).
- These methods work for **descent statistics** only. What is a descent statistic?
- A *descent statistic* is a statistic st such that $st \pi$ depends only on $|\pi|$ and $\text{Des } \pi$ (in other words: if π and σ are two n -permutations with $\text{Des } \pi = \text{Des } \sigma$, then $st \pi = st \sigma$).
Intuition: A descent statistic is a statistic which “factors through Des in each size”.

Compositions & descent compositions: definitions

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A *composition of $n \in \mathbb{N}$* is a composition whose entries sum to n .

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- Let $n \in \mathbb{N}$, and let $[n-1] = \{1, 2, \dots, n-1\}$.

Then, there are mutually inverse bijections

$$\begin{aligned} \text{Des} : \{\text{compositions of } n\} &\rightarrow \{\text{subsets of } [n-1]\}, \\ (i_1, i_2, \dots, i_k) &\mapsto \{i_1 + i_2 + \dots + i_j \mid 1 \leq j \leq k-1\} \end{aligned}$$

and

$$\begin{aligned} \text{Comp} : \{\text{subsets of } [n-1]\} &\rightarrow \{\text{compositions of } n\}, \\ \{s_1 < s_2 < \dots < s_k\} &\mapsto (s_1 - s_0, s_2 - s_1, \dots, s_{k+1} - s_k) \end{aligned}$$

(using the notations $s_0 = 0$ and $s_{k+1} = n$).

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- Let $n \in \mathbb{N}$, and let $[n - 1] = \{1, 2, \dots, n - 1\}$.
Then, there are mutually inverse bijections Des and Comp between $\{\text{subsets of } [n - 1]\}$ and $\{\text{compositions of } n\}$.
If π is an n -permutation, then $\text{Comp}(\text{Des } \pi)$ is called the *descent composition* of π , and is written $\text{Comp } \pi$.

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- **Warning:**

$$\text{Des}((1, 5, 2) \text{ the composition}) = \{1, 6\};$$

$$\text{Des}((1, 5, 2) \text{ the permutation}) = \{2\}.$$

Same for other statistics! Context must disambiguate.

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- inv is not a descent statistic: The permutations $(2, 1, 3)$ and $(3, 1, 2)$ have the same descents, but different numbers of inversions.

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- **However:** Every LR-shuffle-compatible statistic is a descent statistic.

(Better yet, every head-graft-compatible statistic is a descent statistic.)

Quasisymmetric functions, part 1: definition

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- A formal power series $f \in \mathbb{Q}[[x_1, x_2, x_3, \dots]]$ is said to be *quasisymmetric* if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.
- For example:
 - Every symmetric power series is quasisymmetric.
 - $\sum_{i < j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + \cdots$ is quasisymmetric, but not symmetric.

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- Consider the ring $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ of formal power series in countably many indeterminates.
- A formal power series f is said to be *bounded-degree* if the monomials it contains are bounded (from above) in degree.
- A formal power series $f \in \mathbb{Q}[[x_1, x_2, x_3, \dots]]$ is said to be *quasisymmetric* if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.
- For example:
 - Every symmetric power series is quasisymmetric.
 - $\sum_{i < j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + \cdots$ is quasisymmetric, but not symmetric.
- Let **QSym** be the set of all quasisymmetric bounded-degree power series in $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$. This is a \mathbb{Q} -subalgebra, called the *ring of quasisymmetric functions* over \mathbb{Q} . (Gessel, 1980s.)

- For every composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, define

$$M_\alpha = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k}$$

= sum of all monomials whose nonzero exponents are $\alpha_1, \alpha_2, \dots, \alpha_k$ in **this** order.

This is a homogeneous power series of degree $|\alpha|$ (the *size* of α , defined by $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_k$).

- Examples:

- $M_{()} = 1.$

- $M_{(1,1)} = \sum_{i < j} x_i x_j = x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + \dots$

- $M_{(2,1)} = \sum_{i < j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \dots$

- $M_{(3)} = \sum_i x_i^3 = x_1^3 + x_2^3 + x_3^3 + \dots$

Quasisymmetric functions, part 2: the monomial basis

- For every composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, define

$$M_\alpha = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k}$$

= sum of all monomials whose nonzero exponents are $\alpha_1, \alpha_2, \dots, \alpha_k$ in **this** order.

This is a homogeneous power series of degree $|\alpha|$ (the *size* of α , defined by $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_k$).

- The family $(M_\alpha)_\alpha$ is a composition is a basis of the \mathbb{Q} -vector space QSym, called the *monomial basis* (or *M-basis*).

- For every composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, define

$$\begin{aligned}
 F_\alpha &= \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ i_j < i_{j+1} \text{ for all } j \in \text{Des } \alpha}} x_{i_1} x_{i_2} \cdots x_{i_n} \\
 &= \sum_{\substack{\beta \text{ is a composition of } n; \\ \text{Des } \beta \supseteq \text{Des } \alpha}} M_\beta, \quad \text{where } n = |\alpha|.
 \end{aligned}$$

This is a homogeneous power series of degree $|\alpha|$ again.

- Examples:

- $F_{()} = 1.$
- $F_{(1,1)} = \sum_{i < j} x_i x_j = x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + \dots$
- $F_{(2,1)} = \sum_{i \leq j < k} x_i x_j x_k.$
- $F_{(3)} = \sum_{i \leq j \leq k} x_i x_j x_k.$

- For every composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, define

$$\begin{aligned}
 F_\alpha &= \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ i_j < i_{j+1} \text{ for all } j \in \text{Des } \alpha}} x_{i_1} x_{i_2} \cdots x_{i_n} \\
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 \end{aligned}$$

This is a homogeneous power series of degree $|\alpha|$ again.

- The family $(F_\alpha)_{\alpha \text{ is a composition}}$ is a basis of the \mathbb{Q} -vector space QSym , called the *fundamental basis* (or *F-basis*). Sometimes, F_α is also denoted L_α .

- For every composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, define

$$\begin{aligned}
 F_\alpha &= \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ i_j < i_{j+1} \text{ for all } j \in \text{Des } \alpha}} x_{i_1} x_{i_2} \cdots x_{i_n} \\
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 \end{aligned}$$

This is a homogeneous power series of degree $|\alpha|$ again.

- What connects QSym with shuffles of permutations is the following fact:

Theorem. If π and σ are two disjoint permutations, then

$$F_{\text{Comp } \pi} \cdot F_{\text{Comp } \sigma} = \sum_{\tau \in S(\pi, \sigma)} F_{\text{Comp } \tau}.$$

The kernel criterion for shuffle-compatibility

- If st is a descent statistic, then two compositions α and β are said to be *st-equivalent* if $|\alpha| = |\beta|$ and $st \alpha = st \beta$.
(Remember: $st \alpha$ means $st \pi$ for any permutation π satisfying $Comp \pi = \alpha$.)

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- The *kernel* \mathcal{K}_{st} of a descent statistic st is the \mathbb{Q} -vector subspace of QSym spanned by all differences of the form $F_\alpha - F_\beta$, with α and β being two st -equivalent compositions:

$$\mathcal{K}_{st} = \langle F_\alpha - F_\beta \mid |\alpha| = |\beta| \text{ and } st \alpha = st \beta \rangle_{\mathbb{Q}}.$$

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- **Theorem.** The descent statistic st is shuffle-compatible if and only if \mathcal{K}_{st} is an ideal of QSym .
(This is essentially due to Gessel & Zhuang.)

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- The *kernel* \mathcal{K}_{st} of a descent statistic st is the \mathbb{Q} -vector subspace of $QSym$ spanned by all differences of the form $F_\alpha - F_\beta$, with α and β being two st -equivalent compositions:

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- **Theorem.** The descent statistic st is shuffle-compatible if and only if \mathcal{K}_{st} is an ideal of $QSym$.
(This is essentially due to Gessel & Zhuang.)
- Since Ep_k is shuffle-compatible, its kernel \mathcal{K}_{Ep_k} is an ideal of $QSym$. How can we describe it?
- Two ways: using the F -basis and using the M -basis.

The kernel \mathcal{K}_{Epk} in terms of the F -basis

- If $J = (j_1, j_2, \dots, j_m)$ and K are two compositions, then we write $J \rightarrow K$ if there exists an $\ell \in \{2, 3, \dots, m\}$ such that $j_\ell > 2$ and $K = (j_1, j_2, \dots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \dots, j_m)$. (In other words, we write $J \rightarrow K$ if K can be obtained from J by “splitting” some non-initial entry $j_\ell > 2$ into two consecutive entries 1 and $j_\ell - 1$.)
- **Example.** Here are all instances of the \rightarrow relation on compositions of size ≤ 5 :

$$\begin{aligned}(1, 3) &\rightarrow (1, 1, 2), & (1, 4) &\rightarrow (1, 1, 3), \\(1, 3, 1) &\rightarrow (1, 1, 2, 1), & (1, 1, 3) &\rightarrow (1, 1, 1, 2), \\(2, 3) &\rightarrow (2, 1, 2).\end{aligned}$$

- **Proposition.** The ideal \mathcal{K}_{Epk} of QSym is spanned (as a \mathbb{Q} -vector space) by all differences of the form $F_J - F_K$, where J and K are two compositions satisfying $J \rightarrow K$.

The kernel \mathcal{K}_{Epk} in terms of the M -basis

- If $J = (j_1, j_2, \dots, j_m)$ and K are two compositions, then we write $J \xrightarrow[M]{} K$ if there exists an $\ell \in \{2, 3, \dots, m\}$ such that $j_\ell > 2$ and $K = (j_1, j_2, \dots, j_{\ell-1}, 2, j_\ell - 2, j_{\ell+1}, j_{\ell+2}, \dots, j_m)$. (In other words, we write $J \xrightarrow[M]{} K$ if K can be obtained from J by “splitting” some non-initial entry $j_\ell > 2$ into two consecutive entries 2 and $j_\ell - 2$.)
- **Example.** Here are all instances of the $\xrightarrow[M]{}$ relation on compositions of size ≤ 5 :

$$\begin{aligned}(1, 3) &\xrightarrow[M]{} (1, 2, 1), & (1, 4) &\xrightarrow[M]{} (1, 2, 2), \\(1, 3, 1) &\xrightarrow[M]{} (1, 2, 1, 1), & (1, 1, 3) &\xrightarrow[M]{} (1, 1, 2, 1), \\(2, 3) &\xrightarrow[M]{} (2, 2, 1).\end{aligned}$$

- **Proposition.** The ideal \mathcal{K}_{Epk} of QSym is spanned (as a \mathbb{Q} -vector space) by all sums of the form $M_J + M_K$, where J and K are two compositions satisfying $J \xrightarrow[M]{} K$.

- **Question.** Do other descent statistics allow for similar descriptions of \mathcal{K}_{st} ?
(See the paper for some experimental results.)

What does LR-shuffle-compatibility mean algebraically?

- If shuffle-compatible descent statistics induce ideals of QSym , then what do LR-shuffle-compatible descent statistics induce?

(shuffle-compatible des. statistics) \leftrightarrow ((some) ideals of QSym);

(LR-shuffle-compatible des. statistics) \leftrightarrow ??

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- We will answer this question using the *dendriform algebra* structure on QSym .

What does LR-shuffle-compatibility mean algebraically?

- We will answer this question using the *dendriform algebra* structure on QSym .

This structure first appeared in:

Darij Grinberg, *Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions*, *Canad. J. Math.* 69 (2017), pp. 21–53.

But the ideas go back to:

- Glânffrwd P. Thomas, *Frames, Young tableaux, and Baxter sequences*, *Advances in Mathematics*, Volume 26, Issue 3, December 1977, Pages 275–289.
- Jean-Christophe Novelli, Jean-Yves Thibon, *Construction of dendriform trialgebras*, arXiv:math/0510218.

Something similar also appeared in: Aristophanes Dimakis, Folkert Müller-Hoissen, *Quasi-symmetric functions and the KP hierarchy*, *Journal of Pure and Applied Algebra*, Volume 214, Issue 4, April 2010, Pages 449–460.

Dendriform structure on $\mathbb{Q}\text{Sym}$, part 1

- For any monomial m , let $\text{Supp } m$ denote the set $\{i \mid x_i \text{ appears in } m\}$.
- **Example.** $\text{Supp}(x_3^5 x_6 x_8) = \{3, 6, 8\}$.

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- **Example.** $\text{Supp}(x_3^5 x_6 x_8) = \{3, 6, 8\}$.
- We define a binary operation \prec on the \mathbb{Q} -vector space $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ as follows:

- On monomials, it should be given by

$$m \prec n = \begin{cases} m \cdot n, & \text{if } \min(\text{Supp } m) < \min(\text{Supp } n); \\ 0, & \text{if } \min(\text{Supp } m) \geq \min(\text{Supp } n) \end{cases}$$

for any two monomials m and n .

- It should be \mathbb{Q} -bilinear.
- It should be continuous (i.e., its \mathbb{Q} -bilinearity also applies to infinite \mathbb{Q} -linear combinations).
- Well-definedness is pretty clear.
- **Example.** $(x_2^2 x_4) \prec (x_3^2 x_5) = x_2^2 x_3^2 x_4 x_5$, but $(x_2^2 x_4) \prec (x_2^2 x_5) = 0$.

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- We define a binary operation \succ on the \mathbb{Q} -vector space $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ as follows:

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- **Example.** $(x_2^2 x_4) \succ (x_3^2 x_5) = 0$, but $(x_2^2 x_4) \succ (x_2^2 x_5) = x_2^4 x_4 x_5$.

- We now have defined two binary operations \prec and \succ on $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$. They satisfy:

$$a \prec b + a \succ b = ab;$$

$$(a \prec b) \prec c = a \prec (bc);$$

$$(a \succ b) \prec c = a \succ (b \prec c);$$

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- This says that $(\mathbb{Q}[[x_1, x_2, x_3, \dots]], \prec, \succ)$ is a *dendriform algebra* in the sense of Loday (see, e.g., [Zinbiel, *Encyclopedia of types of algebras 2010*, arXiv:1101.0267](#)).

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- \mathcal{QSym} is closed under both operations \prec and \succ . Thus, \mathcal{QSym} becomes a dendriform subalgebra of $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$.

The kernel criterion for LR-shuffle-compatibility

- Recall the **Theorem**: The descent statistic st is shuffle-compatible if and only if \mathcal{K}_{st} is an ideal of $\mathbb{Q}\text{Sym}$.

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- Similarly, **Theorem**: The descent statistic st is LR-shuffle-compatible if and only if

$$QSym \prec \mathcal{K}_{st} \subseteq \mathcal{K}_{st} \quad \text{and} \quad \mathcal{K}_{st} \prec QSym \subseteq \mathcal{K}_{st} \quad \text{and}$$
$$QSym \succeq \mathcal{K}_{st} \subseteq \mathcal{K}_{st} \quad \text{and} \quad \mathcal{K}_{st} \succeq QSym \subseteq \mathcal{K}_{st}$$

(that is, \mathcal{K}_{st} is an ideal of the **dendriform** algebra $QSym$).

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(that is, \mathcal{K}_{st} is an ideal of the **dendriform** algebra $QSym$).

- Thus, for example, \mathcal{K}_{Epk} is an ideal of the **dendriform** algebra $QSym$, and the quotient $QSym / \mathcal{K}_{Epk}$ is a dendriform algebra.

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(that is, \mathcal{K}_{st} is an ideal of the **dendriform** algebra $QSym$).

- Thus, for example, \mathcal{K}_{Epk} is an ideal of the **dendriform** algebra $QSym$, and the quotient $QSym / \mathcal{K}_{Epk}$ is a dendriform algebra.
- This actually inspired the (combinatorial) proof of LR-shuffle-compatibility hinted at above.

A few questions

- **Question.** What mileage do we get out of \mathcal{Z} -enriched (P, γ) -partitions for other choices of \mathcal{N} and \mathcal{Z} than the ones used in the known proofs?
- **Question.** What ring do the $K_{n, \Lambda}^{\mathcal{Z}}$ span?
- **Question.** Hsiao and Petersen have generalized enriched (P, γ) -partitions to “colored (P, γ) -partitions” (with $\{+, -\}$ replaced by an m -element set). Does this generalize our results?
- **Question.** How do the kernels \mathcal{K}_{st} look like for $st = \text{Pk}, \text{Lpk}, \dots$?
- **Question.** Are the quotients $\text{QSym} / \mathcal{K}_{st}$ for $st = \text{des}, \text{Lpk}, \text{Epk}$ known dendriform algebras?

Section 4

Quadri-compatibility (work in progress)

References:

- a forthcoming preprint.
- Marcelo Aguiar, Jean-Louis Loday, *Quadri-algebras*, Journal of Pure and Applied Algebra, Volume 191 (2004), Issue 3, Pages 205–221.
- Loïc Foissy, *Free quadri-algebras and dual quadri-algebras*, arXiv:1504.06056.

- We can refine LR-shuffle-compatibility even further.
- Given two disjoint nonempty permutations $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ and $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_m)$, define sets $S_{i,j}(\pi, \sigma)$ for all $i, j \in \{1, 2\}$ as follows:

$$S_{1,1}(\pi, \sigma) = \{\tau \in S(\pi, \sigma) \mid \tau_1 = \pi_1 \text{ and } \tau_{n+m} = \pi_n\};$$

$$S_{1,2}(\pi, \sigma) = \{\tau \in S(\pi, \sigma) \mid \tau_1 = \pi_1 \text{ and } \tau_{n+m} = \sigma_m\};$$

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- A statistic st is said to be *quadri-compatible* if for any two disjoint nonempty permutations π and σ and any $i, j \in \{1, 2\}$, the multiset

$$\{st \tau \mid \tau \in S_{i,j}(\pi, \sigma)\}_{\text{multiset}}$$

depends only on $st \pi$, $st \sigma$, $|\pi|$, $|\sigma|$, i, j , the truth value of $\pi_1 > \sigma_1$, and the truth value of $\pi_n > \sigma_m$.

- A permutation statistic st is said to be *tail-graft-compatible* if for any nonempty permutation $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ and any letter a that does not appear in π , the element $st(\pi : a)$ depends only on $st(\pi)$, $|\pi|$ and on the truth value of $a > \pi_n$. Here, $\pi : a$ is the permutation obtained from π by appending a at the end:

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- **(Almost-)Theorem (G.)** A statistic st is quadri-compatible **if and only if** it is shuffle-compatible, head-graft-compatible and tail-graft-compatible.
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- Hence, Des, des, and Epk are quadri-compatible. (But not maj or Lpk or Rpk or Pk.)

- A permutation statistic st is said to be *tail-graft-compatible* if for any nonempty permutation $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ and any letter a that does not appear in π , the element $st(\pi : a)$ depends only on $st(\pi)$, $|\pi|$ and on the truth value of $a > \pi_n$. Here, $\pi : a$ is the permutation obtained from π by appending a at the end:

$$\pi = (\pi_1, \pi_2, \dots, \pi_n) \quad \Longrightarrow \quad \pi : a = (a, \pi_1, \pi_2, \dots, \pi_n, a).$$

- **(Almost-)Theorem (G.)** A statistic st is quadri-compatible **if and only if** it is shuffle-compatible, head-graft-compatible and tail-graft-compatible.
- My proof uses both induction and QSym and still needs to be written up. (Hopefully it survives the process.)
- Hence, Des, des, and Epk are quadri-compatible. (But not maj or Lpk or Rpk or Pk.)
- The proof (so far) uses a refined version of dendriform algebras: the *quadri-algebras* of Aguiar and Loday ([arXiv:math/0309171](https://arxiv.org/abs/math/0309171), [arXiv:1504.06056](https://arxiv.org/abs/1504.06056)).

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Thank you for attending!

slides: <http://www.cip.ifi.lmu.de/~grinberg/algebra/dartmouth18.pdf>

paper: <http://www.cip.ifi.lmu.de/~grinberg/algebra/gzshuf2.pdf>

project: <https://github.com/darijgr/gzshuf>