Pattern Avoidance in Motzkin Paths

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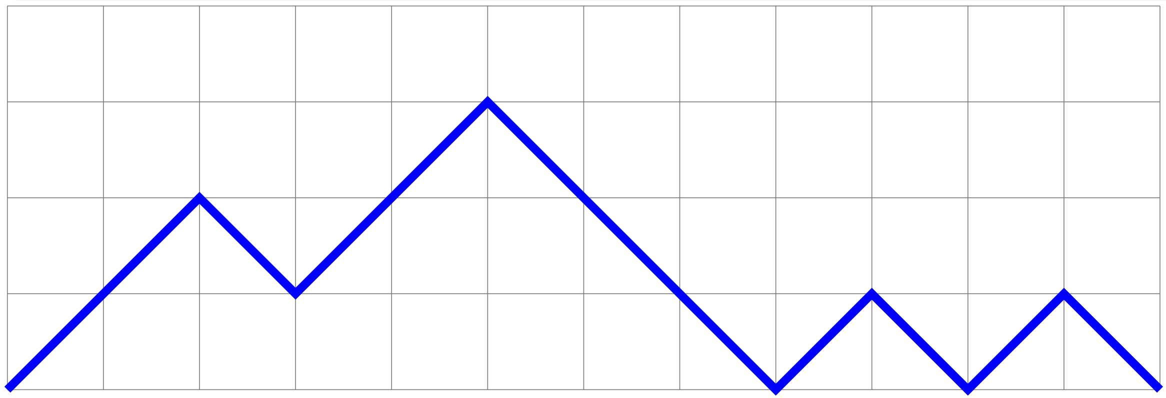
Outline

Dyck Paths

Definition

A *Dyck Path* of semilength n is a lattice path from $(0,0)$ to $(2n,0)$ allowing $(1,1)$ and $(1,-1)$ steps never going below the x-axis.



UUDUUDDDUDUD

Patterns in Dyck Paths

Bernini, Ferrari, Pinzani, West. (2013)

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Definition

A Dyck path π contains a pattern σ if π contains σ as a subword. Otherwise π avoids σ.

UUDUUD contains *UDUD*, but avoids *UUUDDD*.

Notation: If σ is a pattern, will use σ^k to denote $\underbrace{\sigma\sigma\ldots\sigma}_{k \text{ times}}$.

Patterns in Dyck Paths

Let P be a set of patterns. $D_n(P)$ is the set of all Dyck paths of semilength n avoiding all elements of P and $d_n(P) = \#D_n(P)$.

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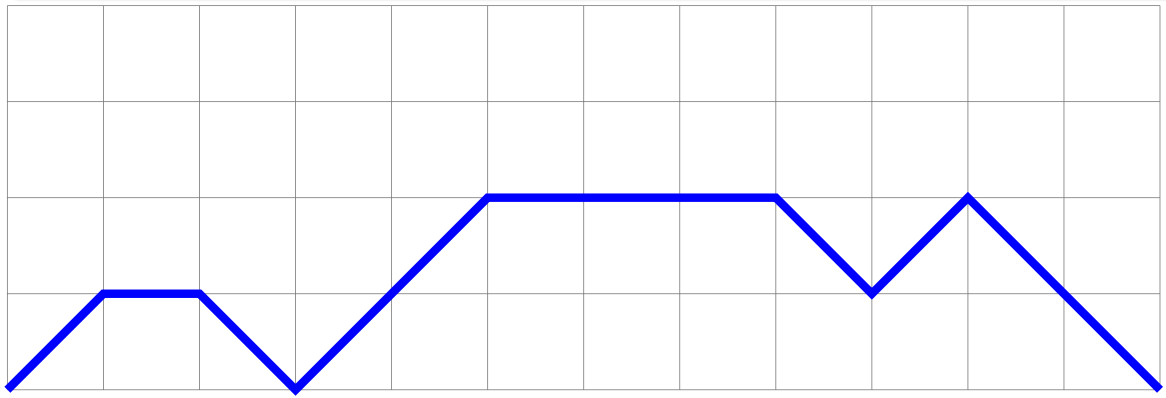
Theorem (Bernini et al, 2013)

- $d_n(UD) = 0$
- $d_n((UD)^2) = 1$
- $d_n((UD)^3) = 1 + \binom{n}{2}$
- $d_n((UD)^k) = \sum_{i=0}^{k-1} N_{n,i}$ *($N_{n,i} = n, i^{th}$ Narayana number)*

Motzkin Paths

Definition

A *Motzkin path* of length n is a lattice path from $(0,0)$ to $(n,0)$ allowing $(1,1)$, $(1,-1)$ and $(1,0)$ steps never going below the x-axis.



UHDUUHHHDUDD

Motzkin Paths and Motzkin Numbers

Motzkin paths are counted by Motzkin numbers (OEIS A001006), M_n.

n	1	2	3	4	5	6	7	8	9	10
m_n	1	2	4	9	21	51	127	323	835	2188

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Define avoidance, $M_n(P)$ and $m_n(P)$ analogously with that of Dyck paths.

Pattern of Length 1

Pattern of Length 1: H

Pattern of Length 1

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Theorem

- $m_{2n}(H) = C_n$
- $m_{2n+1}(H) = 0$

Patterns of Length 2

Patterns of Length 2: UD, H^2

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Theorem

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- $m_{2n}(H^2) = C_n$
- $m_{2n+1}(H^2) = (2n+1)C_n$

Patterns of Length 2

Patterns of Length 2: UD, H^2

Theorem

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- $m_{2n+1}(H^2) = (2n+1)C_n$

OEIS(A057977) - Alois P. Heinz

n	1	2	3	4	5	6	7	8	9	10	11	12
$m_n(H^2)$	1	1	3	2	10	5	35	14	126	42	462	132

Patterns of Length 3

There are **four** patterns of length 3: UDH, HUD, UHD, H^3.

There are **three** Wilf-equivalence classes:
$\{UDH, HUD\}, \{UHD\}, \{H^3\}$.

UDH and HUD

Theorem (D., Ramey)

$m_n(UDH) = m_n(HUD)$

Proof.

Given any $\pi \in M_n(UDH)$, reverse π and switch all U's and D's. □

Construction of UDH recurrence

- Consider a path $\pi \in M_n(UDH)$. This path can either begin with an H or a U.

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$$U\underline{\hspace{3cm}}D\underline{\hspace{3cm}}$$

Construction of UDH recurrence

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$$U\_\_\_\_\_\_\_\_\_\_ D\_\_\_\_\_\_\_\_\_\_$$

- Assuming D is in position i, there are $m_{i-2}(UDH)$ paths between the U and D and to the right of the D must be a Dyck path.

UDH and HUD

Theorem (D., Ramey)

$$m_n(UDH) = m_{n-1}(UDH) + \sum_{i=2}^{n} m_{i-2}(UDH) C_{(n-i)/2}$$

where $C_{\frac{n-i}{2}}$ is Catalan if $n-i$ is even and 0 otherwise.

UDH and HUD

Theorem (D., Ramey)

$$m_n(UDH) = m_n(HUD) = \binom{n}{\lfloor n/2 \rfloor} \qquad (A001405)$$

n	1	2	3	4	5	6	7	8	9	10	11	12
$m_n(UDH)$	1	2	3	6	10	20	35	70	126	252	462	924

A corollary

Corollary

$$\binom{n}{\lfloor n/2 \rfloor} = \binom{n-1}{\lfloor (n-1)/2 \rfloor} + \sum_{i=2}^{n} \binom{i-2}{\lfloor (i-2)/2 \rfloor} C_{(n-i)/2}$$

UHD

Build a similar recurrence.

- If $\pi \in M_n(UHD)$ starts with H, then attach any such path of length $n-1$. There are $m_{n-1}(UHD)$ such paths.

UHD

Build a similar recurrence.

- If $\pi \in M_n(UHD)$ starts with H, then attach any such path of length $n-1$. There are $m_{n-1}(UHD)$ such paths.
- If π starts with U, then consider the *last* D in the path.

$$U\_\_\_\_\_\_\_\_\_\_\_D\_\_\_\_\_\_\_\_\_\_\_$$

UHD

Build a similar recurrence.

- If $\pi \in M_n(UHD)$ starts with H, then attach any such path of length $n-1$. There are $m_{n-1}(UHD)$ such paths.
- If π starts with U, then consider the *last* D in the path.

$$U\underline{\hspace{4cm}}D\underline{\hspace{4cm}}$$

- In between the U and D must be a Dyck path. After the D must be all H's.

$$m_n(UHD) = m_{n-1}(UHD) + \sum_{i=1}^{\lfloor n/2 \rfloor} C_i$$

UHD

Theorem (D., Ramey)

$$m_n(UHD) = 1 + \sum_{i=1}^{\lfloor n/2 \rfloor} (n - 2i + 1) \cdot C_i$$

n	1	2	3	4	5	6	7	8	9	10	11	12
$m_n(UHD)$	1	2	3	6	9	17	25	47	69	133	197	393

Not in OEIS

H^3

Either there are 0 H's, 1 H, or 2 H's.

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Path of even length: 0 H's or 2 H's.
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Theorem (D., Ramey)

- $m_{2n}(H^3) = C_n + \binom{2n}{2} C_{n-1}$
- $m_{2n+1}(H^3) = (2n+1)C_n$

H^k

Theorem (D., Ramey)

$$m_{2n}(H^k) = \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{2n}{2i} C_{n-i}$$

$$m_{2n+1}(H^k) = \sum_{i=0}^{\lfloor (k-2)/2 \rfloor} \binom{2n+1}{2i+1} C_{n-i}$$

$(UD)^2$

Theorem (D., Ramey)

$$m_n((UD)^2) = 2^{n-1}$$

U^2D^2

Theorem (D., Ramey)

$$m_n(U^2D^2) = 1 + \binom{n}{2} + \binom{n}{4}$$

U^2D^2

Theorem (D., Ramey)

$$m_n(U^2D^2) = 1 + \binom{n}{2} + \binom{n}{4}$$

n	1	2	3	4	5	6	7	8	9	10	11	12
$m_n(U^2D^2)$	1	2	4	8	16	31	57	99	163	256	386	562

Future Work

- Patterns of length 4: *HUDH*, *UDHH*, *HHUD*
- UH^kD, $(UD)^k$, U^kD^k
- Connections between these pattern avoiding paths and ideals of certain affine Lie algebras.

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Thank you!

References

- A. Bernini, L. Ferrari, R. Pinzani, J. West. *Pattern-Avoiding Dyck Paths*. Alain Goupil and Gilles Schaeffer. 25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013), 2013, Paris, France. p. 683-694.
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