

Rook and Wilf equivalence of integer partitions

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(Joint work with Dan Saracino & Nathan McNew)

Permutation Patterns – Dartmouth College

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Some basic definitions

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$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_h > 0$$

with $|\lambda| = \lambda_1 + \dots + \lambda_h = n$.

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★ We identify partitions with **Ferrers diagrams**. E.g.,

$$4+3+2+1 = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \\ \square & \square & & \\ \square & & & \end{array}$$

$$3+3+3 = \begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array}$$

$$1+1+1+1 = \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array}$$

$$4 = \begin{array}{cccc} \square & \square & \square & \square \end{array}$$

Rook Theory

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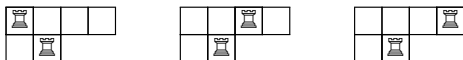
Question: How many configurations of k non-attacking rooks can be placed on a partition (Ferrers diagram)?



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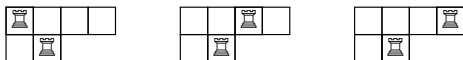
For any partition $\mu \in \mathbb{P}$ we define its **rook polynomial** to be

$$R_\mu(q) = \sum_{k \geq 0} r(\mu, k) q^k$$

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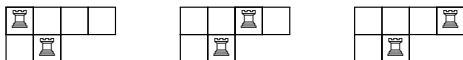
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Two partitions $\mu, \tau \in \mathbb{P}$ are **rook equivalent** provided that

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★ μ, τ admit the same number of k -configurations.

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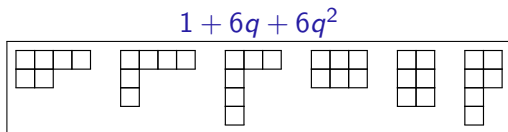
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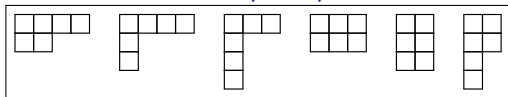
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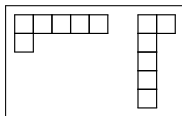
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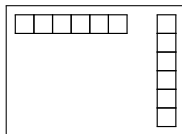
$$1 + 6q + 6q^2$$



$$1 + 6q + 4q^2$$



$$1 + 6q$$



$$1 + 6q + 7q^2 + q^3$$



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Theorem (Foata & Schützenberger - 1970)

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- (i) μ and τ are rook equivalent
- (ii) We have equality of the “L-multisets”

$$\{1 + \mu_1, 2 + \mu_2, 3 + \mu_3, \dots\} = \{1 + \tau_1, 2 + \tau_2, 3 + \tau_3, \dots\}$$

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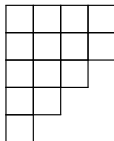
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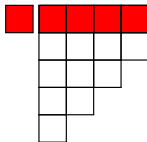
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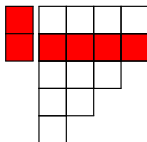
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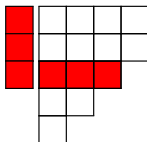
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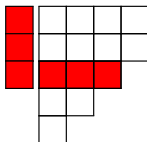
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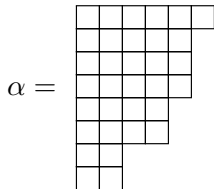
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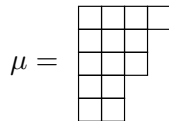
► Hence the term *L*-multisets

Integer partition patterns

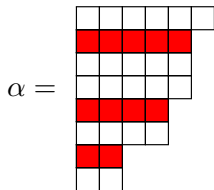
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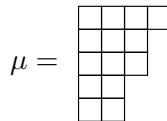
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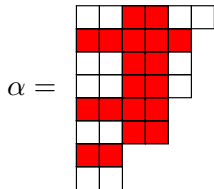
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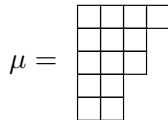
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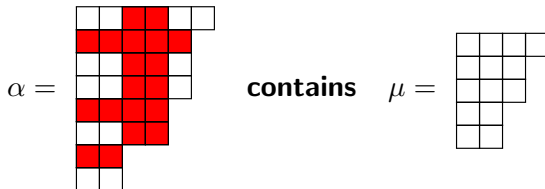
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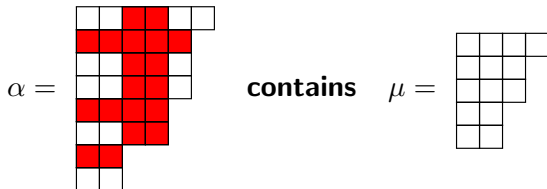
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Definition (Bloom, Saracino)

We say α **contains** μ if one can delete rows and columns from α to obtain μ .

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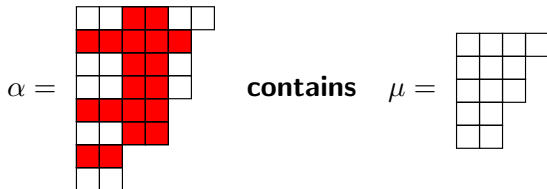


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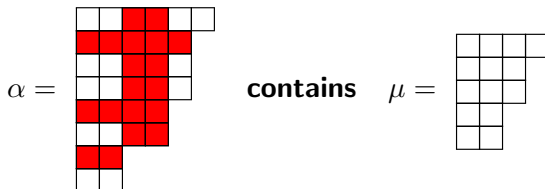
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- ▶ $\mathbb{P}_n(\mu, k) = \{\alpha \in \mathbb{P}_n(\mu) \mid \alpha_1 = \mu_1 + k\}$

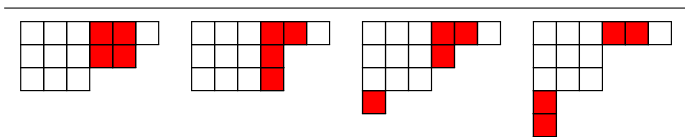
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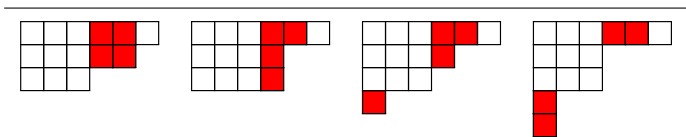
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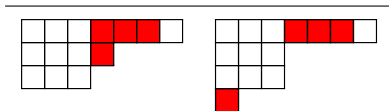
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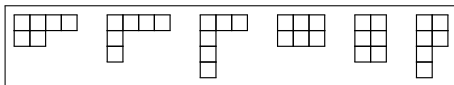
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Refining this, μ, τ are **width-Wilf equivalent** provided

$$P_{\mu,k}(q) = P_{\tau,k}(q) \quad (\text{for all } k \geq 0)$$

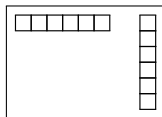
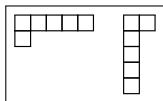
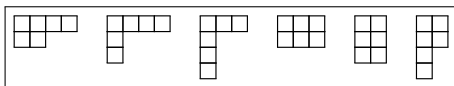
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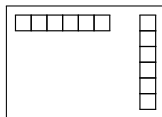
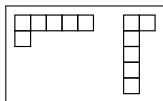
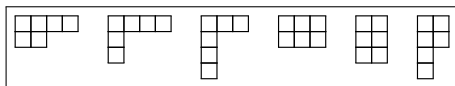
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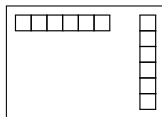
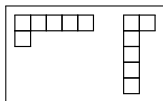
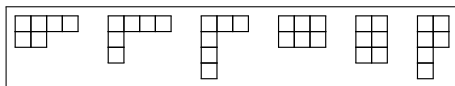
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(\Rightarrow) Characterize $P_{\mu,k}(q)$ in terms of “L-multisets”.

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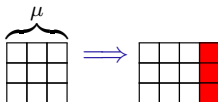
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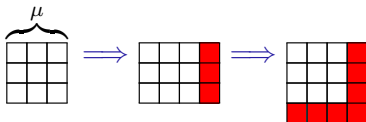
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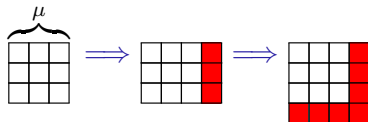
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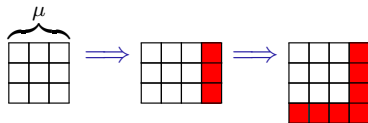
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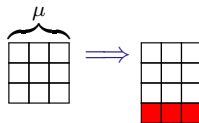
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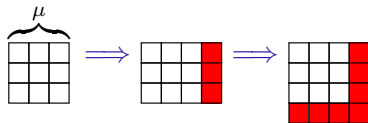
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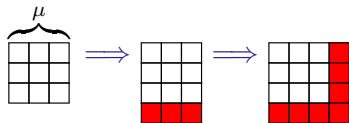
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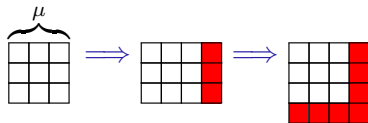
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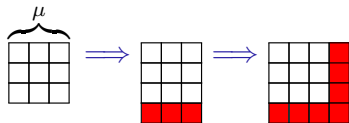
Proof sketch: rook \implies Wilf

Idea: Build $\mathbb{P}(\mu)$ by inserting rows/columns into μ .

Problem: Multiple ways to build same partition!!



or



★ Need Inclusion/Exclusion!

Proof sketch: $\text{rook} \implies \text{Wilf}$

Proof sketch: rook \implies Wilf

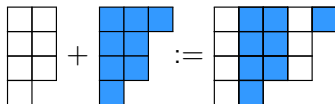
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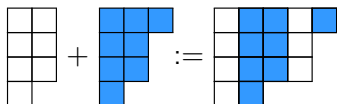


★ An operation on columns

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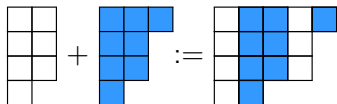
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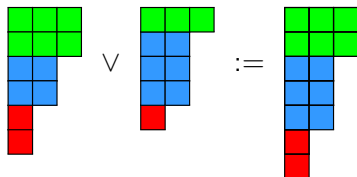
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★ An operation on rows

Proof sketch: rook \implies Wilf

Definition

For any (finite) $S \subset \mathbb{P}$ we define $\vee_S : \mathbb{P} \rightarrow \mathbb{P}$ as

$$\vee_S(\mu) := \bigvee_{\alpha \in S} (\mu + \alpha)$$

Proof sketch: rook \implies Wilf

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S ranges over sets of partitions with k columns.

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 \implies Characterize when the operators

$$\vee_S = \vee_T$$

for subsets S and T .

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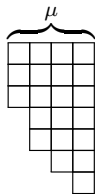
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Proof sketch: rook \implies Wilf

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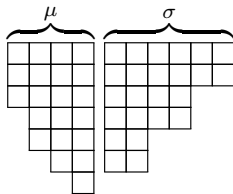
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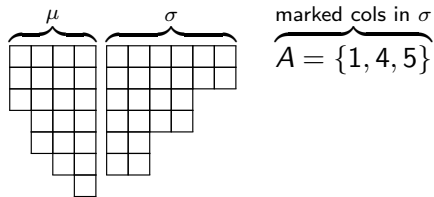
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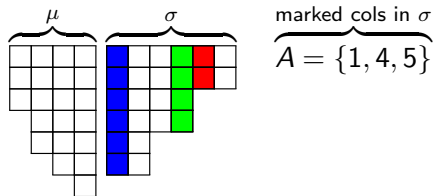
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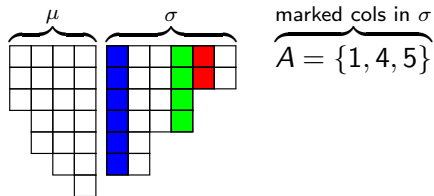
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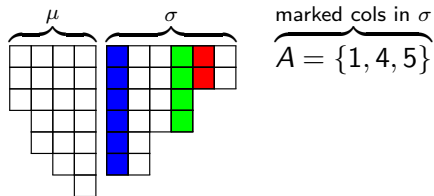


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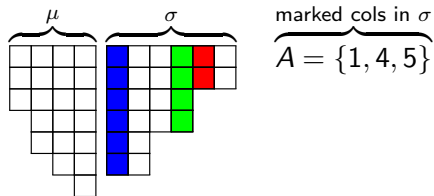
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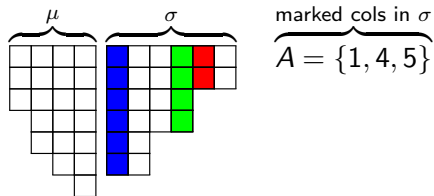
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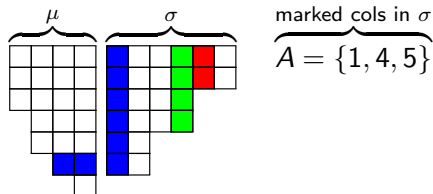
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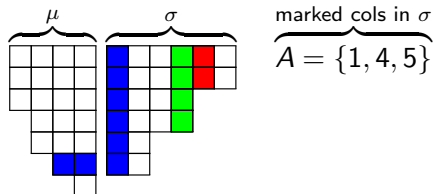
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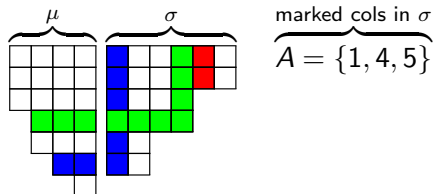
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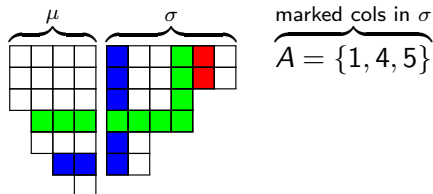
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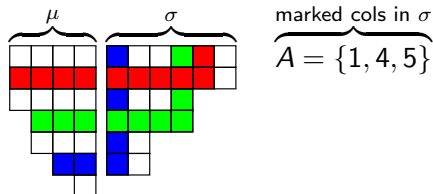
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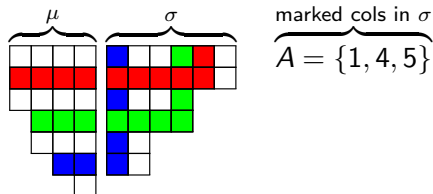
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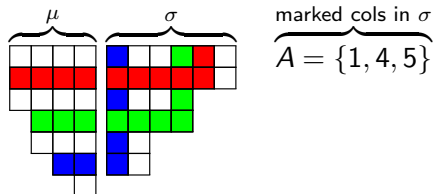
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★ Marked structures “index” L -multisets!

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after loads of cancellation...

$$P_{\mu,k}(q) = \frac{q^{|\mu|}}{(1-q) \cdots (1-q^{k+\mu_1})} \sum_{(\mu, \sigma, A) \in \mathcal{M}(\mu, k)} (-1)^{|A|} q^{|\mu, \sigma, A|}$$

Enumeration results

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Let $\mu \in \mathbb{P}$ so that $|\mu_i - \mu_j| > 1$ for $i \neq j$, then

$$\sum_{n \geq 0} |\mathbb{P}_n \setminus \mathbb{P}_n(\mu)| q^n$$

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$\implies \tau$ avoids $\mu = (N + 1, 1)$

Thank You!