Pattern Avoidance in Rooted Trees

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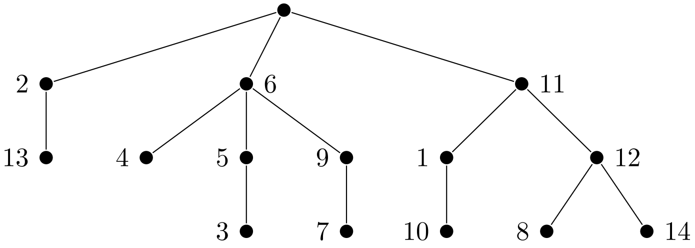
Trees

We are working with rooted, labeled, unordered trees on $[n]$ with the root left unlabeled.



Tree Pattern Avoidance

A rooted labeled tree on $[n]$ avoids the pattern $\sigma \in S_k$ if along each path from the root to a vertex, the sequence of labels does not contain a subsequence that is in the same relative order as σ.

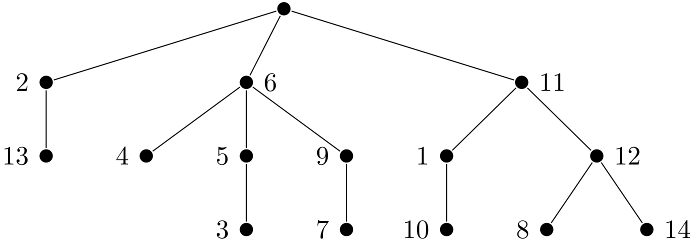


Figure: For example, this tree avoids 213.

Motivation

Previous results:

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- A nice formula exists for (unrooted) alternating trees (Postnikov, 1997) and a generating function for the rooted analog (Kuba, Panholzer, 2008)
- Several other statistics on trees have been studied, including number of leaves, proper vertices, vertex-descents, vertex-inversions, etc.

Wilf-classes

Proposition

For $n \geq 1$, $t_n(\sigma_1, \ldots, \sigma_m) = t_n(\sigma_1^c, \ldots, \sigma_m^c)$.

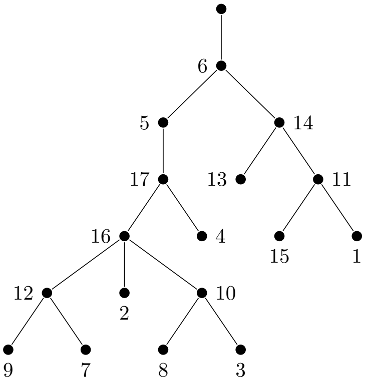
	Classical			Consecutive		
n	321	231	132	321	231	132
1	1	1	1	1	1	1
2	3	3	3	3	3	3
3	15	15	15	15	15	15
4	104	104	104	107	106	106
5	918	917	918	997	973	972

Figure: The number of trees on $[n]$ that avoid the given pattern.

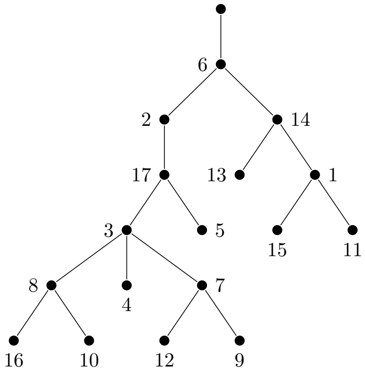
Wilf-classes

Theorem

For all $n \geq 1$, $t_n(312) = t_n(321)$

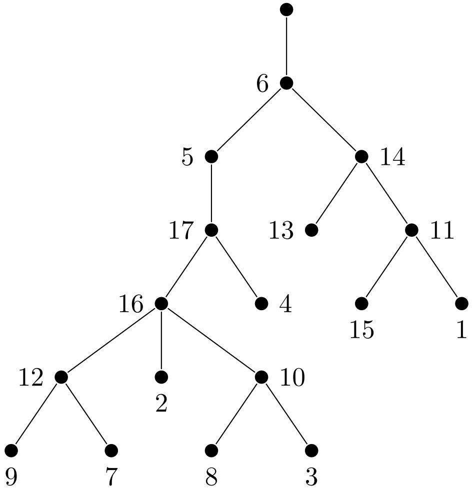


Avoids 312



Avoids 321

Bijection: $T_n(312) \to T_n(321)$



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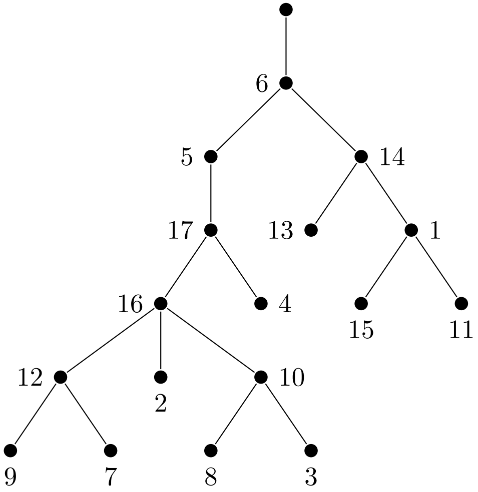

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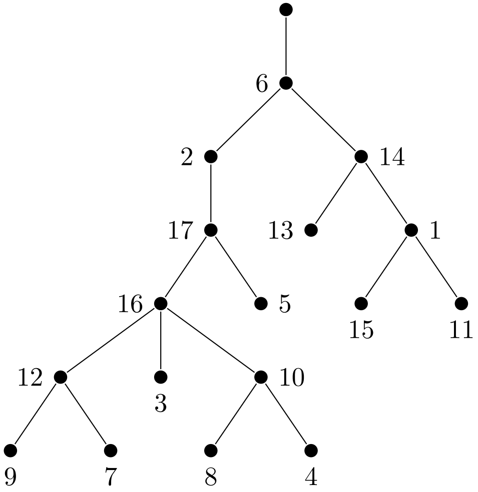

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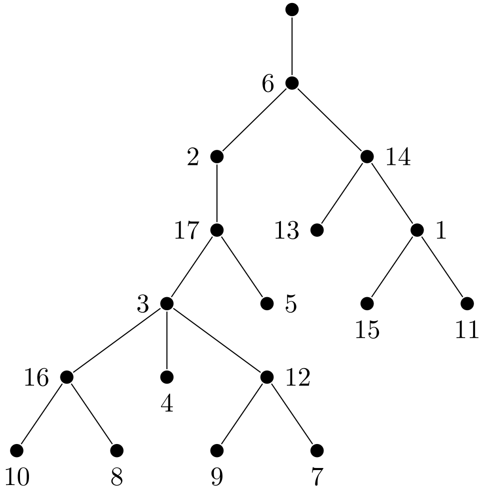

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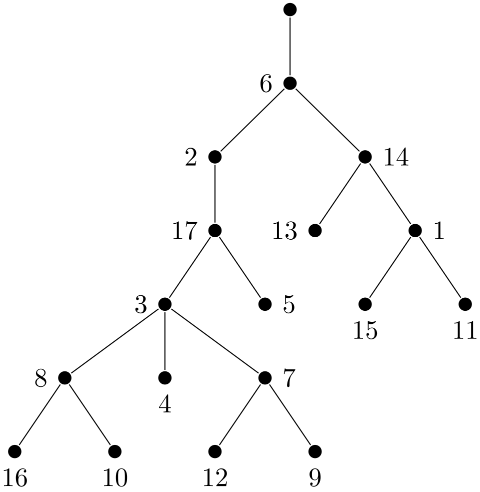


Bijections

Avoiding	Enumeration	Bijection
21	$n!$	permutations
213, 312	$\sum_{k=1}^{n} k!c(n,k)$	ordered cycle decompositions
213, 312, 321	$\sum_{k=1}^{n} k!S(n,k)$	ordered partitions
213, 312, 123	$\sum_{k=1}^{n} \mathcal{B}(k)\, c(n,k)$	partitioned cycle decompositions
312, 213, 132 231, 132, 213	$\sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}$	partitions into ordered lists
321, 2143, 3142	$n! + \sum \frac{n!}{2^{\ell}}\binom{n-k-1}{\ell-1}\binom{k}{\ell}$	ordered partitions into ordered lists (up to reverse)

Theorem

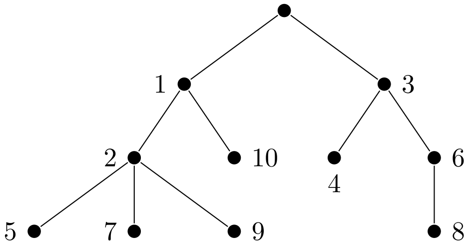
There are $n!$ increasing trees on $[n]$. (I.e. There are $n!$ trees that avoid 21.)

Proof.

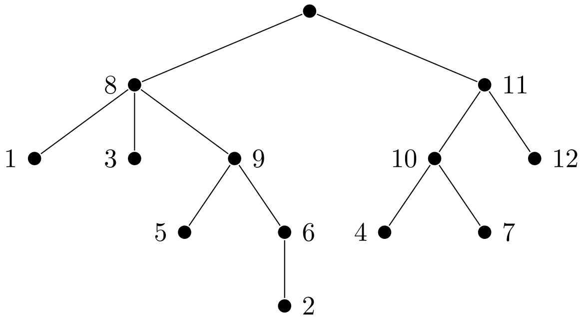
In *Enumerative Combinatorics, Vol. I*, Stanley provides a bijection ϕ from permutations on $[n]$ to increasing trees on $[n]$. □

Given $\pi \in S_n$, the tree $\varphi(\pi)$ is obtained by letting all left-right minima be the children of the root. For the remaining vertices, let i be the child of the rightmost element j of π that precedes i and is less than i.

Example: $\varphi(3, 6, 8, 4, 1, 10, 2, 9, 7, 5)$



Unimodal Trees



Along each path from the root to any vertex, the list of labels is increasing then decreasing. In other words, it avoids 213 and 312.

Unimodal Trees

Theorem

The number of trees that avoid 213 and 312 on $[n]$ is

$$\sum_{k=1}^{n} k!c(n,k)$$

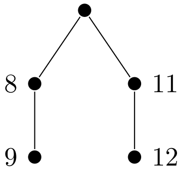
where $c(n,k)$ is the unsigned Stirling numbers of the first kind, i.e. the number of permutations in S_n which can be decomposed into k disjoint cycles.

Proof.

Construct a bijection θ from the set of ordered cycle decompositions of permutations in S_n to unimodal trees. □

$\theta((11, 4, 10, 7)(12)(8, 3, 1)(9, 5, 2, 6))$

The ordered list of maxima is $11, 12, 8, 9$. Use φ to get an increasing tree from this sequence.



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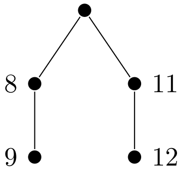
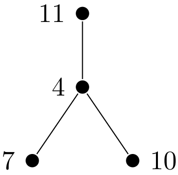
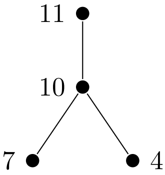


The elements in each cycle which are not the maximum are the descendants of that maximum in the tree. Use φ on the remaining elements of each cycle and then take the complement to obtain a decreasing tree on those elements.

$(11, 4, 10, 7) \xrightarrow{\varphi}$



$\xrightarrow{c}$



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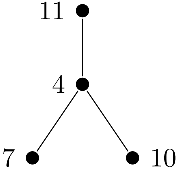

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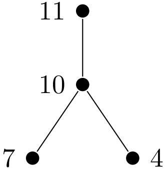
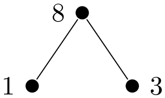


$(8, 3, 1) \xrightarrow{\varphi}$



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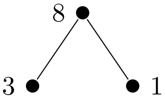

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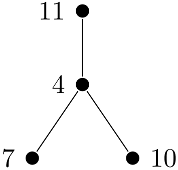

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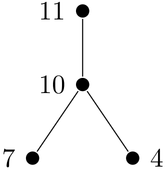

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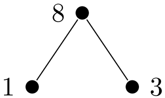

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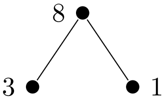
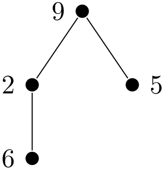


$(9, 5, 2, 6) \xrightarrow{\varphi}$



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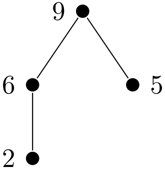
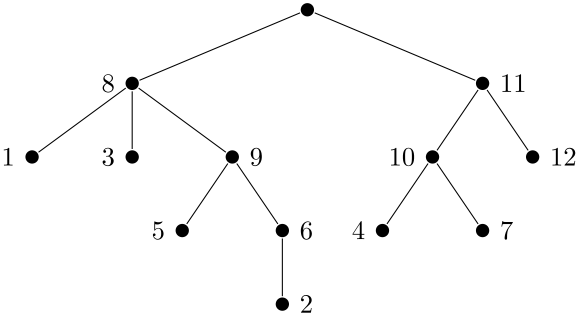


$\theta((11,4,10,7)(12)(8,3,1)(9,5,2,6))$



Unimodal trees that avoid 321

Theorem

The number of trees that avoid 213, 312, and 321 is

$$\sum_{k=1}^{n} k!S(n,k),$$

where $S(n,k)$ is the Stirling numbers of the second kind, i.e. the number of ways to partition the set $[n]$ into k non-empty subsets.

Proof.

Construct a bijection γ from the set of ordered partitions of $[n]$ to unimodal trees that avoid 321. □

Example

Given an ordered partition of $[n]$ into k parts, the ordered list of maxima from each part determines the "increasing" part of the tree. The elements in each part which are not the maximum are the children of that maximum in the tree.

- Ordered partition of [12] : $\{2, 5, 7, 9\}, \{4, 10\}, \{6, 8, 11\}, \{1, 3\}, \{12\}$

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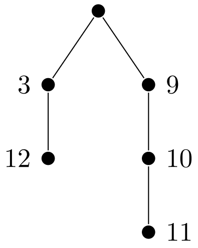
- Ordered partition of $[12]$: $\{2,5,7,9\},\{4,10\},\{6,8,11\},\{1,3\},\{12\}$
- Ordered list of maxima: $9, 10, 11, 3, 12$



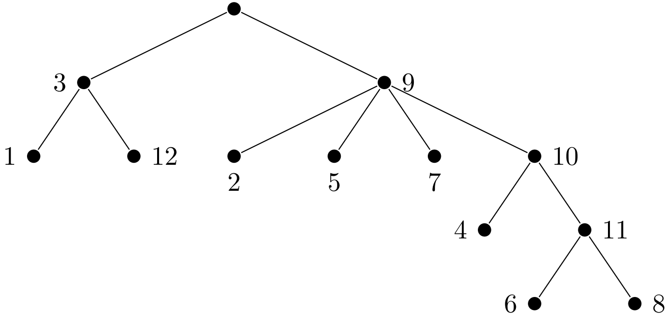


Figure: $\gamma(\{9, 2, 5, 7\}, \{4, 10\}, \{11, 6, 8\}, \{1, 3\}, \{12\})$

THANK YOU!

For more information, see:
Rooted forests that avoid sets of permutations
arXiv: 1607.03046