Wilf-collapse in permutation classes

Michael Albert (University of Otago)

joint work with
Vít Jelínek and Michal Opler

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UNIVERSITY of OTAGO
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Te Whare Wānanga o Otāgo
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Definitions

Let $\mathcal{C}$ be a permutation class.

- The *enumeration sequence* for $\mathcal{C}$ is just $(c_n)_{n \geqslant 0}$ where c_n is the number of permutations of size n in $\mathcal{C}$
- For $\pi \in \mathcal{C}$, the *principal subclass*, $\mathcal{C}_\pi = \mathcal{C} \cap Av(\pi)$ of $\mathcal{C}$ consists of all those permutations in $\mathcal{C}$ that *do not* have π as a subpermutation.

Wilf-equivalence and Wilf-collapse

- Two permutation classes $\mathcal{C}$ and $\mathcal{D}$ are *Wilf-equivalent* if $c_n = d_n$ **for all** n.
- If $\mathcal{C}$ is a class, and $\pi, \tau \in \mathcal{C}$ then π is *Wilf-equivalent* to τ relative to $\mathcal{C}$ ($\pi \sim_{\mathcal{C}} \tau$) if the principal subclasses $\mathcal{C}_\pi$ and $\mathcal{C}_\tau$ are Wilf-equivalent.
- $\mathcal{C}$ exhibits a *Wilf-collapse* if the number, w_n, of $\sim_{\mathcal{C}}$-equivalence classes for permutations of size n in $\mathcal{C}$ is small compared to c_n, i.e., $w_n = o(c_n)$.

Some prior work

- (2015, MA and Mathilde Bouvel) *The class* Av(312) *has an exponential Wilf-collapse. In fact*

$$w_n = o((5/8)^n c_n).$$

  This bound is not tight.
- (2017, MA and Jinge Li) *Any class with two basis elements of size three has an (exponential if possible) Wilf-collapse.*
- In the first case we have a conjecturally complete description of $\sim_{\mathcal{C}}$, in the second case a complete description.
- If there is a theme it is: *global symmetry applied locally often results in Wilf-equivalence*.

A dream and a plan

- The dream is to show that many classes exhibit Wilf-collapse.
- The plan is to find sufficient conditions for Wilf-equivalence that apply to a wide variety of classes.
- For example if we could show that: *except for exponentially few permutations in $\mathcal{C}$ their $\sim_{\mathcal{C}}$-equivalence classes are exponentially large*, that gets an exponential Wilf-collapse.

A particular case

- Consider classes $\mathcal{C}$ which contain only a finite set, $\mathcal{A}$, of sum-indecomposable permutations.
- The elements of $\mathcal{C}$ are naturally represented as words over the alphabet $\mathcal{A}$ via their sum-decomposition.
- This is a bijection between $\mathcal{C}$ and $\mathcal{A}^*$ if $\mathcal{C}$ is sum-closed.
- The subpermutation ordering on $\mathcal{C}$ is a refinement of the subword ordering on $\mathcal{A}^*$ which can be thought of as taking the transitive closure of it and certain basic relations $A \leqslant a$ where $A = a_1 a_2 \cdots a_k \in \mathcal{A}^*$, $a \in \mathcal{A}$, and

$$a_1 \oplus a_2 \oplus \cdots \oplus a_k \leqslant a$$

as permutations.

Gekko's principle

If,

- $\mathcal{A}$ is a set of sum-indecomposable permutations.
- $A \in \mathcal{A}^*$ and $B = bC \in \mathcal{A}^*$
- $A = PS$ where P is the maximal prefix of A with $P \leqslant b$.

Then,

$$A \leqslant B \iff S \leqslant C.$$

The basic idea

- Let $\mathcal{C} = \oplus\mathcal{A}$
- Let a be a maximal element of $\mathcal{A}$.
- Consider two words:

$$\begin{array}{lclcccccc} A & = & X & a(\bullet)a & Y & a({}^{\bullet}{}_{\bullet})a & Z \\ B & = & X & a({}^{\bullet}{}_{\bullet})a & Y & a(\bullet)a & Z \end{array}$$

- Claim: $A \sim_{\mathcal{C}} B$.
- Reason: If $A \leqslant W$ witness this greedily. Now exchange the "part of W that matches the $(\bullet)$" and the "part of W that matches the $({}^{\bullet}{}_{\bullet})$" to produce a W' with $B \leqslant W'$ and note this is a bijection.
- The key point is that because of the maximality of a it can only be matched by literally some other a so it forces a break in the word.

Why this helps

- Words over a finite weighted alphabet have good statistical behaviour (an instance of the *supercritical* case discussed in F&S).
- In particular, in a word of weight n, the number of occurrences of any fixed subword (e.g., $a\,(\bullet)\,a$) is linear (in n) except with exponentially small probability.
- From previous slide, any two words which differ only in the order of their $a\,(\bullet)\,a$ or $a\,({}^{\bullet}{}_{\bullet})\,a$ factors are Wilf-equivalent.
- And

$$\binom{(c+d)n}{cn}$$

  grows exponentially.

A slight generalisation

- Let $\mathcal{C}$ be a sum-closed class whose sum-indecomposables are $\mathcal{A}$, with generating function $A(t)$.
- Suppose that $A(t) = 1$ has a solution r strictly inside its radius of convergence.
- Suppose there exist $a, b \in \mathcal{A}$ such that $ab \not\leq c$ for all $c \in \mathcal{A}$.

Theorem

$\mathcal{C}$ has an exponential Wilf-collapse.

Note: *None* of these conditions apply to $\mathcal{C} = \mathrm{Av}(312)$ but the conclusion holds.

Is sum-closure needed?

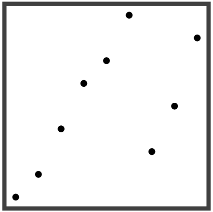
- No.
- In the original setting above it is the case that for some $k \geq 0$ and some $c \geq 1$,

$$c_n = \Theta(n^k c^n).$$

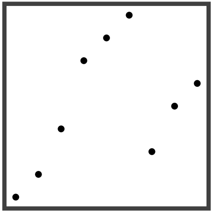
- A finite state automaton recognises $\mathcal{C}$.
- Almost all paths in this automaton of weight n visit a special kind of state exactly $k+1$ times.
- Those typical paths belong to exponentially large Wilf-classes.
- It's not clear that the residue is exponentially small so we do not yet have an exponential Wilf-collapse.
- This argument *may* generalise to the supercritical case provided that $\mathcal{C}$ is finitely based.

A word of warning

In Av(321) the *shortest* pair of permutations beginning with their two least elements that appear to be Wilf-equivalent[1] but not for reasons of symmetry are:



$12 \oplus 2457136$



$12 \oplus 2567134$

However, it has not been ruled out that $\alpha \oplus 2457136 \sim_{\mathrm{Av}(321)} \alpha \oplus 2567134$, for all α, which would be a positive indication of eventual Wilf-collapse in Av(321).

[1] checked to size 18

Lessons learned

- A greedy method for checking for involvement lets one apply ‘global’ Wilf-equivalences to ‘local’ pieces.
- This generally demonstrates an exponential Wilf-collapse.
- In retrospect, arguments of MA and Mathilde Bouvel in the class of 312-avoiding permutations can also be cast in this light.
- We have many other examples of lesser and greater specificity – but they all fit into that framework.
- In *every* example where $c_n \to \infty$ and we can compute or estimate w_n it is the case that $w_n = o(c_n)$.